Section 3.1: Introduction to sequences and series

Question 50

Does it make sense to talk about the "number"

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots ?$$

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- $1 + \frac{1}{4} = 1.25$

$$\frac{\pi^2}{6} \approx 1.644934$$

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- $1 + \frac{1}{4} = 1.25$
- $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.423611$

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- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10)^2} \approx 1.549767$

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- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10000)^2} \approx 1.644834$

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- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10000)^2} \approx 1.644834$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(100000)^2} \approx 1.644924$

$$\frac{\pi^2}{6} \approx 1.644934$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

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converges to the number $\frac{\pi^2}{6}$ (we will have precise definitions for the highlighted terms a bit later).

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converges to the number $\frac{\pi^2}{6}$ (we will have precise definitions for the highlighted terms a bit later).

This fact is remarkable - there is no obvious connection between π and squares of the form $\frac{1}{n^2}$; moreover all the terms in the series are rational but $\frac{\pi^2}{6}$ is certainly not.
This example gives us in principle a way of calculating the digits of π or at least of π^2. (In practice there are similar but better ways, as the convergence in this example is very slow).

Another Example

Example 51

What about

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots ?$$

Try experimenting with initial segments again :

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50} \approx 4.4992$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

Another Example

Example 51

What about

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots ?$$

Try experimenting with initial segments again :

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50} \approx 4.4992$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100} \approx 5.1874$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

Another Example

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What about

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots ?$$

Try experimenting with initial segments again :

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50} \approx 4.4992$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100} \approx 5.1874$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1000} \approx 7.4855$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

Another Example

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What about

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...?$$

Try experimenting with initial segments again :

- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50} \approx 4.4992$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100} \approx 5.1874$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1000} \approx 7.4855$
- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50000} \approx 11.3970$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

Another Example . . .

Example 52

What about

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ...?$$

Experimenting reveals

- $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$
- $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} = \frac{341}{1024} \approx 0.33301$
- $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots + \frac{1}{2^{14}} \approx 0.3333$

These calculations can be verified directly using properties of sums of geometric progressions. It appears that this series is converging (quite fast) to $\frac{1}{3}$.

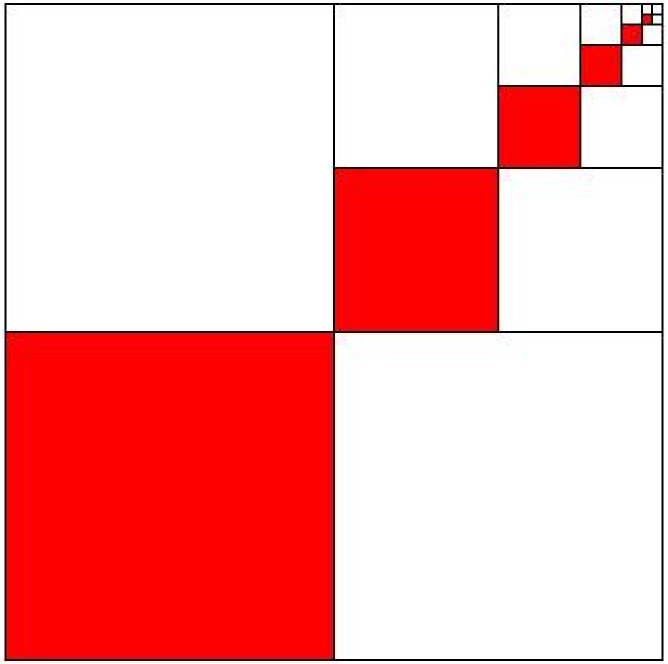
Another Example ...

Example 53

What about

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots ?$$

The following picture gives some graphical evidence for this hypothesis.



A last example

Example 54

Does it make sense to talk about

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

as a function of x?

If it does, then f must have a domain (consisting of some or all of the real numbers?) and substituting these values in to the definition in place of x must somehow make sense.

- $x = 0$: $f(0) = 0$

In all cases we get (just from the first six terms) something very close to $\sin x$.

A last example

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- $x = 0$: $f(0) = 0$
- $x = \frac{\pi}{2}$: $f(\frac{\pi}{2}) \approx 0.9999$ (six terms)

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A last example

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- $x = 0$: $f(0) = 0$
- $x = \frac{\pi}{2}$: $f(\frac{\pi}{2}) \approx 0.9999$ (six terms)
- $x = \frac{\pi}{6}$: $f(\frac{\pi}{6}) \approx 0.5000$ (six terms)

In all cases we get (just from the first six terms) something very close to $\sin x$.

A last example

Example 54

Does it make sense to talk about

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

as a function of x*?*

If it does, then f must have a domain (consisting of some or all of the real numbers?) and substituting these values in to the definition in place of x must somehow make sense.

- $x = 0$: $f(0) = 0$
- $x = \frac{\pi}{2}$: $f(\frac{\pi}{2}) \approx 0.9999$ (six terms)
- $x = \frac{\pi}{6}$: $f(\frac{\pi}{6}) \approx 0.5000$ (six terms)
- $x = \frac{\pi}{3}$: $f(\frac{\pi}{3}) \approx 0.8660$ (six terms) ($\frac{\sqrt{3}}{2} \approx 0.8660$)

In all cases we get (just from the first six terms) something very close to $\sin x$.

Section 3.2 : Sequences

Note: Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

Definition 55

A sequence is an infinite ordered list

$$a_1, a_2, a_3, \dots$$

- The items in list a_1, a_2 etc. are called terms (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence $a_1, a_2, \dots$ can be denoted by (a_n) or by $(a_n)_{n=1}^{\infty}$.
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty}$: $a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0, \; a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \dots$

$$0, 2, 0, 2, 0, 2, \dots$$

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   $$0, 2, 0, 2, 0, 2, \dots$$

2. $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \sin(\pi) = 0,\ a_3 = \sin(\frac{3\pi}{2}) = -1,\ a_4 = \sin(2\pi) = 0, \dots$.

   $$1, 0, -1, 0, 1, 0, -1, 0, \dots$$

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty}$: $a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0,\ a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \dots$

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   $$1, 0, -1, 0, 1, 0, -1, 0, \dots$$

3. $(\frac{1}{n}\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \frac{1}{n}\sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \frac{1}{2}\sin(\pi) = 0,\ a_3 = \frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3},\ a_4 = \frac{1}{4}\sin(2\pi) = 0, \dots$.

   $$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \dots$$