MA203/MA283: LINEAR ALGEBRA
SEMESTER 2 2024-25
PRACTICE PROBLEM SHEET 3

1. Determine whether each of the following subsets of $\mathbb{R}^3$ is linearly independent.

   (a) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \right\}$

   (b) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \right\}$

2. Let S be a linearly independent subset of a vector space V, and suppose that v is an element of V with $v \notin \langle S \rangle$. Show that $S \cup \{v\}$ is a linearly independent subset of V.

3. Extend the set
$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \right\}$$
to a basis of $\mathbb{R}^3$.

4. Show that $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

5. What is the dimension of the space of all symmetric matrices in $M_3(\mathbb{R})$?
(Recall that A is symmetric if $A^T = A$).

6. Let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$. What is the dimension of the space $v^\perp$ defined by
$$v^\perp = \{u \in \mathbb{R}^3 : u^T v = 0\}?$$
Find a basis of $u^\perp$.

7. Find the change of basis matrix from the standard basis to the basis B of Question 4 above, and use it to find the B-coordinates of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (where the elements of B are ordered as in Question 4).

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(v) = Av$, where A is the matrix $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -2 & 3 \\ -1 & 0 & -2 \end{bmatrix}$.
What is the matrix of T with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$?

9. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v$.
   Show that there is no basis of $\mathbb{R}^2$ with respect to which the matrix of T is diagonal.

   (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v$.
   Show that there is no basis of $\mathbb{R}^2$ with respect to which the matrix of T is diagonal.

10. Find a matrix P for which $P^{-1}AP = \mathrm{diag}(1, -2, -10)$, where
$$A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 0 & 4 \\ -4 & 2 & -10 \end{bmatrix}$$
(Note that you do not need to calculate the characteristic polynomial of A to answer this!)

11. Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -2 \\ 0 & -1 & 2 \end{bmatrix}$, and hence find the eigenvalues of A. Find an eigenvector corresponding to each eigenvalue, and determine whether A is diagonalizable.

12. Give an example of a 2×2 matrix whose entries are all non-zero integers and whose eigenvalues are 2 and 4.