Lecture 15: Rank, Coordinates and Change of Basis

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Recall from Lecture 14

1. A basis of a vector space is a spanning set that is linearly independent.
   - Spanning set means that every element of the vector space is a linear combination of the elements in the basis.
   - Linearly independent means that no proper subset of the basis is sufficient to be a spanning set of the space.
2. If a vector space has a finite spanning set, it is called finite dimensional. Then it has a finite basis.
3. In a finite dimensional vector space, every basis has the same number of elements. This number is the dimension of the space.

Row rank and column rank

Let A be a $m \times n$ matrix.

The row rank r of A is the dimension of the row space of A, which is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

The column rank c of A is the dimension of the column space of A, which is the subspace of $\mathbb{R}^m$ spanned by the columns of A. The column rank is the dimension of the image of the linear transformation whose matrix is A.

The row rank is at most m and the column rank is at most n, but both can be less.

Theorem The row rank and column rank are same for every matrix.

So we can just refer to the rank.

Row rank = Column rank

For a $m \times n$ matrix A, write r for its row rank and c for its column rank.

Choose a basis for the row space of A, and write its elements as the rows of a $r \times n$ matrix P.

Since every row of A is a linear combination of the rows of P, there is a $m \times r$ matrix Q for which $A = QP$. (Think about this!)

But now every column of A is a linear combination of the r columns of Q, so the dimension of the column space of A is at most r: $c \leq r$.

To see that $r \leq c$ also, start with a basis for the column space of A, and write its c elements as the columns of a $m \times c$ matrix P'. Then $A = P'Q'$ for some $c \times n$ matrix Q', and every row of A is a linear combination of the c rows of Q'. Hence $r \leq c$.

Since $c \leq r$ and $r \leq c$, we conclude $r = c$.

Coordinates

Lemma If $\{b_1, \ldots, b_n\}$ is a basis of a vector space V, then every element of V has a unique expression as a linear combination of $b_1, \ldots, b_m$.

Proof Suppose for some $v \in V$ that

$$\begin{aligned} v &= a_1 b_1 + a_2 b_2 + \ldots a_n b_n, \text{ and} \\ v &= a'_1 b_1 + a'_2 b_2 + \ldots a'_n b_n, \end{aligned}$$

for scalars a_i, a'_i. Then $0_V = (a_1 - a'_1)b_1 + (a_2 a'_2)b_2 + \cdots + (a_n - a'_n)b_n$. Since B is linearly independent, the coefficients $a_i - a'_i$ are all zero and the two expressions for v are identical. $\square$

Example In $\mathbb{R}^2$, the standard coordinates of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ are $(4, 3)$. With respect to the basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$, the coordinates of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ are $(2, 1)$. This is saying that

$$\boxed{\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix}.}$$

Coordinates with respect to different bases

Let B be the (ordered) basis of $\mathbb{R}^3$ with elements

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix},\ b_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},\ b_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

Question Given an element of $\mathbb{R}^3$, for example $v = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$, how do we find the B-coordinates of v?

Answer We know $v = 2e_1 - 3e_2 + 4e_3$. We need the B-coordinates of the standard basis vectors e_1, e_2, e_3. Then $[v]_B = 2[e_1]_B - 3[e_2]_B + 4[e_3]_B$.[1]
To find $[e_1]_B$:

$$e_1 = xb_1 + yb_2 + zb_3 \Longrightarrow \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} z = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is saying that $[e_1]_B$ is the first column of the inverse of the matrix that has the elements of the basis B as its three columns. Columns 2 and 3 of this inverse are $[e_2]_B$ and $[e_3]_B$.

[1] We write $[v]_B$ for the column vector with the B-coordinates of v as entries

Change of Basis

Let B be the (ordered) basis of $\mathbb{R}^3$ with elements

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix},\ b_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},\ b_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

How to find the B-coordinates of $v \in \mathbb{R}^3$, for example $v = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$:

Write a 3×3 matrix P whose columns are the basis elements of B.

$$P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ 4 & 0 & 2 \end{bmatrix},\quad P^{-1} = \begin{bmatrix} -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ 0 & -1 & 0 \\ \frac{2}{7} & \frac{4}{7} & -\frac{1}{14} \end{bmatrix}$$

To find the B-coordinates of any $v \in \mathbb{R}^3$, multiply v on the left by P^{-1}.

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}_B = \begin{bmatrix} -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ 0 & -1 & 0 \\ \frac{2}{7} & \frac{4}{7} & -\frac{1}{14} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{12}{7} \\ 3 \\ -\frac{10}{7} \end{bmatrix}$$

This is saying $v = \frac{12}{7} b_1 + 3b_2 - \frac{10}{7} b_3$, which can be checked.

> P^{-1} is called the change of basis matrix from the standard basis to B.

The Rank-Nullity Theorem

The Rank-Nullity Theorem relates the dimensions of the kernel, image and domain of a linear transformation. The dimension of the image of a linear trasformation is called its *rank*, and the dimension of the kernel is called the *nullity*. The rank of T is equal to the rank of the matrix of T, since the image of T is the column space of this matrix.

Theorem (Rank-Nullity Theorem) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, where V and W are finite-dimensional vector spaces over a field $\mathbb{F}$. Then

$$\dim(\ker T) + \operatorname{rank} T = n.$$

Informally, the Rank-Nullity Theorem says that the full dimension of the domain must be accounted for in the combination of the kernel and the image.

Proof of the Rank-Nullity Theorem

$T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation. $\boxed{\dim(\ker T) + \operatorname{rank} T = n.}$

1. Write k for $\dim(\ker T)$ and let $\{b_1, \dots, b_k\}$ be a basis of $\ker T$.
2. Extend this to a basis $\{b_1, \dots, b_k, c_{k+1} \dots, c_n\}$ of $\mathbb{R}^n$.
3. Since T sends each b_i to 0, the image under T of every element of $\mathbb{R}^n$ is a linear combination of $T(c_{k+1}), \dots, T(c_n)$.
4. Also $\{T(c_{k+1}), \dots, T(c_n)\}$ is a linearly independent subset of $\mathbb{R}^m$. To see this, suppose for some scalars $a_{k+1}, \dots, a_n$ that $a_{k+1}T(c_{k+1}) + a_{k+2}T(c_{k+2}) + \cdots + a_nT(c_n) = 0$. Then

   $$a_{k+1}c_{k+1} + \cdots + a_nc_n \in \ker T \Longrightarrow a_{k+1}c_{k+1} + \cdots + a_nc_n \in \langle b_1, \dots, b_k \rangle.$$

   Since $\{b_1, \dots, b_k, c_{k+1} \dots, c_n\}$ is linearly independent in $\mathbb{R}^n$, this means that $a_{k+1}c_{k+1} + a_{k+2}c_{k+2} + \cdots + a_nc_n = 0$, and each $a_j = 0$.
5. It follows that $\{T(c_{k+1}), \dots, T(c_n)\}$ is a basis for the image of T, so this image has dimension $n - k$, as required.