# Lecture 13: Linear Independence and the Replacement Theorem

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# Linear Dependence and Linear Independence

For a subset $\{v_1, \dots, v_k\}$ of $\mathbb{R}^n$, suppose that $v_k$ is a linear combination of $v_1, \dots, v_{k-1}$. Then every linear combination of $v_1, \dots, v_k$ is "already" a linear combination of $v_1, \dots, v_{k-1}$ and

$$\langle v_1, \dots, v_k \rangle = \langle v_1, \dots, v_{k-1} \rangle.$$

If we are interested in the span of $\{v_1, \dots, v_k\}$ we could throw away $v_k$ and this would not change the span.

Definition A set of (at least two) vectors in $\mathbb{R}^n$ is linearly dependent if one of its elements is a linear combination of the others.
A set of vectors in $\mathbb{R}^n$ is linearly independent if it is not linearly dependent.[1]

Linear independence means that throwing away any element results in shrinking the span.

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[1] Small print: a set with just one vector is linearly independent, unless this vector is the zero vector. Any set that contains the zero vector is linearly dependent.

# More on Linear Independence

## Example from Lecture 2

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array} \quad \begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three equations of the system, or the three rows of the original augmented matrix, formed a *linearly dependent set*. One row was eliminated by adding a linear combination of the other two. All the information in the system was contained in just (any) two of the three equations.

The non-zero rows of the reduced row-echelon form are linearly independent, and they span the rowspace of the original matrix. The rowspace is the subspace of $\mathbb{R}^5$ that is spanned by the rows.

# Meaning of linear independence - theory and practice

**Meaning** of linear independence A set is linearly independent if none of its elements is a linear combination of the others.

This definition makes conceptual sense, but to use it as a test for linear independence would mean checking it separately for every element of the set - not so efficient. We have an alternative formulation for this purpose, which is logically equivalent but maybe a bit more obscure as a description.

> A set of vectors is linearly independent if the only way to write the zero vector as a linear combination of its elements is by taking all the coefficients to be zero.

# Test for linear independence

To decide if the set $\{v_1, \dots, v_k\}$ is linearly independent, try to write the zero vector as a linear combination of the $v_i$:

$$\sum_{i=1}^{k} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0,$$

for scalars $a_1, \dots, a_k$. If $a_i = 0$ for every $i$ is the only solution, then $v_1, \dots, v_k$ are linearly independent. If there is another solution, they are linearly dependent.

Example Decide whether $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^3$. Solution By row reduction we find

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies a = b = c = 0.$$

Conclusion The set is linearly independent.

# Finite dimensional spaces

Definition A vector space $V$ is finite dimensional if it contains a finite spanning set.

This means a set $\{v_1, \dots, v_k\}$ of elements, with the property that every element of $V$ is a linear combination of $v_1, \dots, v_k$.

Examples

1. $\mathbb{R}^n$ is finite dimensional, with $\{e_1, \dots, e_n\}$ as a spanning set with $n$ elements.
2. $M_{m\times n}(\mathbb{R})$ is finite dimensional, with $\{E_{ij}\}_{1\leq i\leq m,\ 1\leq j\leq n}$ as a spanning set with $mn$ elements. The matrix $E_{ij}$ has 1 in position $(i, j)$ and zero in all other positions.
3. An example of vector space that is not finite dimensional is $\mathbb{R}[x]$, the space of all polynomials with coefficients in $\mathbb{R}$. If $S$ is any finite set of polynomials, then the degree of a linear combination of elements of $S$ can't exceed the highest degree of a polynomial in $S$.

# What is a basis?

A **basis** of a vector space is a linearly independent spanning set.

- A basis is a minimal spanning set, one in which every element is needed, one that does not contain a smaller spanning set.
- Example: $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$.
  In general $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.
- $\{(1,3),(1,4)\}$ is a basis of $\mathbb{R}^2$.
- If $S$ is a finite spanning set of a vector space $V$, then $S$ contains a basis of $V$. If $S$ is not linearly independent, then some $v \in S$ is a linear combination of the other elements of $S$. Throwing $v$ away leaves a smaller set that still spans $V$. Repeat this step until a basis remains.

# The Steinitz Replacement Lemma

**Theorem** Let $V$ be a vector space that has a basis with $n$ elements. Then every linearly independent set with $n$ elements is a basis of $V$.

**Proof** (for $n = 3$). Suppose $B = \{b_1, b_2, b_3\}$ is a basis of $V$, and let $\{y_1, y_2, y_3\}$ be a linearly independent subset of $V$.

1. $y_1 = a_1 b_1 + a_2 b_2 + a_3 b_3$ for scalars $a_1, a_2, a_3$, not all zero.
   We can assume (after maybe relabelling the $b_i$), that $a_1 \neq 0$.
   Then
   $$b_1 = a_1^{-1} y_1 - a_1^{-1} a_2 b_2 - a_1^{-1} a_3 b_3.$$
   So $b_1 \in \langle y_1, b_2, b_3 \rangle$ and $\{y_1, b_2, b_3\}$ spans $V$.

   (Note that we have to use the fact that we can divide by non-zero scalars to write $b_1$ as a linear combination of $y_1, b_2, b_3$.)

# Proof of replacement lemma (continued)

2. Now $y_2 \in \langle y_1, b_2, b_3 \rangle$ and $y_2$ is not a scalar multiple of $y_1$ (because $\{y_1, y_2, y_3\}$ is linearly independent).
   So $b_2$ (or $b_3$) has non-zero coefficient in any description of $y_2$ as a linear combination of $y_1, b_2, b_3$.
   Replace again: $\{y_1, y_2, b_2\}$ spans $V$.
3. Same reasoning: we can replace $b_2$ with $y_3$ to conclude $\{y_1, y_2, y_3\}$ spans $V$.

Conclusion $\{y_1, y_2, y_3\}$ is a basis of $V$.