

Section 1.5 : Improper Integrals

Suppose that $f(x)$ is a continuous function that satisfies

$$\lim_{x \rightarrow \infty} f(x) = 0;$$

for example $f(x) = e^{-x}$ has this property. Then we can consider the total area between the graph $y = f(x)$ and the X -axis, to the right of (for example) $x = 1$. This area is denoted by

$$\int_1^{\infty} f(x) dx$$

and referred to as an **improper integral**. For a given function, it is not clear whether the area involved is finite or infinite (if it is infinite, the improper integral is said to **diverge** or to be **divergent**). One question that arises is how we can determine if the relevant area is finite or infinite, another is how to calculate it if it is finite.

Definition of $\int_a^\infty f(x) dx$

Definition 33

If the function f is continuous on the interval $[a, \infty)$, then the *improper integral* $\int_a^\infty f(x) dx$ is defined by

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

provided this limit exists. In this case the improper integral is called **convergent** (otherwise it's divergent).

Similarly, if f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^a f(x) dx := \lim_{b \rightarrow -\infty} \int_b^a f(x) dx$$

Convergent or Divergent?

So to calculate an improper integral of the form $\int_1^{\infty} f(x) dx$ (for example), we first calculate

$$\int_1^b f(x) dx$$

for a general b . This will typically be an expression involving b . We then take the limit as $b \rightarrow \infty$.

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Example 34

Show that the improper integral $\int_1^{\infty} \frac{1}{x} dx$ is divergent.

Solution:

$$\int_1^b \frac{1}{x} dx = \ln x \Big|_1^b = \ln b - \ln 1 = \ln b.$$

Since $\ln b \rightarrow \infty$ as $b \rightarrow \infty$, $\lim_{b \rightarrow \infty} \ln b$ does not exist and the integral diverges.

Another Example

Example 35

Evaluate $\int_{-\infty}^{-2} \frac{1}{x^2} dx$.

Solution:

$$\int_b^{-2} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_b^{-2} = \frac{1}{2} + \frac{1}{b}$$

Then $\lim_{b \rightarrow -\infty} \left(\frac{1}{2} + \frac{1}{b} \right) = \frac{1}{2}$, and

$$\int_{-\infty}^{-2} \frac{1}{x^2} dx = \frac{1}{2}.$$

Yet another example

Example 36

Determine whether $\int_1^{\infty} e^{-2x} dx$ is convergent or divergent, and evaluate it if it is convergent.

Solution: $\int_1^b e^{-2x} dx = -\frac{1}{2}e^{-2x} \Big|_1^b = -\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2}$. Then

$$\int_1^{\infty} e^{-2x} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2} \right) = \frac{e^{-2}}{2}.$$

So the integral is convergent and the enclosed area is $\frac{1}{2e^2}$.

Another type of improper integral

If the graph $y = f(x)$ has a **vertical asymptote** for a value of x in the interval $[c, d]$, these needs to be considered when computing the integral $\int_c^d f(x) dx$, since this integral describes the area of a region that is infinite in the vertical direction at the asymptote.

- If the vertical asymptote is at the left endpoint c , then we define

$$\int_c^d f(x) dx = \lim_{b \rightarrow c^+} \int_b^d f(x) dx.$$

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- If the vertical asymptote is at the right endpoint d , then we define

$$\int_c^d f(x) dx = \lim_{b \rightarrow d^-} \int_c^b f(x) dx.$$

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If the graph $y = f(x)$ has a **vertical asymptote** for a value of x in the interval $[c, d]$, these needs to be considered when computing the integral $\int_c^d f(x) dx$, since this integral describes the area of a region that is infinite in the vertical direction at the asymptote.

- If the vertical asymptote is at an interior point m of the interval $[c, d]$, then we define

$$\int_c^d f(x) dx = \int_c^m f(x) dx + \int_m^c f(x) dx,$$

and the two improper integrals involving m are handled as above.

As in the case of improper integrals of the first type, these improper integrals are said to *converge* if the limits in question can be evaluated and to *diverge* if these limits do not exist. Divergence means that the area involved is infinite.

Final Example

Example 37

Determine whether the improper integral $\int_{-2}^4 \frac{1}{x^2} dx$ is convergent or divergent.

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Determine whether the improper integral $\int_{-2}^4 \frac{1}{x^2} dx$ is convergent or divergent.

What makes this integral improper is the fact that the graph $y = \frac{1}{x^2}$ has a vertical asymptote at $x = 0$. Thus

$$\int_{-2}^4 \frac{1}{x^2} dx = \int_{-2}^0 \frac{1}{x^2} dx + \int_0^4 \frac{1}{x^2} dx$$

Final Example (continued)

For the first of these two integrals we have

$$\begin{aligned}\int_{-2}^0 \frac{1}{x^2} dx &= \lim_{b \rightarrow 0^-} \int_{-2}^b \frac{1}{x^2} dx \\ &= \lim_{b \rightarrow 0^-} \left(-\frac{1}{x} \right) \Big|_{-2}^b \\ &= \lim_{b \rightarrow 0^-} \left(-\frac{1}{b} + \frac{1}{2} \right)\end{aligned}$$

Since $\lim_{b \rightarrow 0^-} \left(-\frac{1}{b} \right)$ does not exist, the improper integral $\int_{-2}^0 \frac{1}{x^2} dx$ **diverges**. This means that the area enclosed between the graph $y = \frac{1}{x^2}$ and the x -axis over the interval $[-2, 0]$ is **infinite**.

Final Example (continued)

For the first of these two integrals we have

$$\begin{aligned}\int_{-2}^0 \frac{1}{x^2} dx &= \lim_{b \rightarrow 0^-} \int_{-2}^b \frac{1}{x^2} dx \\ &= \lim_{b \rightarrow 0^-} \left(-\frac{1}{x} \right) \Big|_{-2}^b \\ &= \lim_{b \rightarrow 0^-} \left(-\frac{1}{b} + \frac{1}{2} \right)\end{aligned}$$

Now that we know that the first of the two improper integrals in our problem diverges, we don't need to bother with the second. The improper integral $\int_{-2}^4 \frac{1}{x^2} dx$ is **divergent**.

Note of Caution

In this last example, if we had not noticed the vertical asymptote at $x = 0$, we might have proceeded as follows:

$$\begin{aligned}\int_{-2}^4 \frac{1}{x^2} dx &= \left(-\frac{1}{x} \right) \Big|_{-2}^4 \\ &= -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}\end{aligned}$$

This would be wrong! We should check that an integral is not improper before evaluating it in the manner above.

That's the end of Chapter 1

The first exam question (and first two homework assignments) are based on Chapter 1. The purpose of these items is to assess how well you have achieved the learning outcomes for Chapter 1. Section 1.6 of the lecture notes has some exam advice for Chapter 1, including sample “exam-type” questions with worked solutions and answers.

Things to do:

- Make sure you know how to use all the notation involving integrals, and distinguish definite, indefinite and improper integrals.
- Make sure you understand what the Fundamental Theorem of Calculus, so that you can state it and apply it to examples.
- Practise, practise, practise the techniques of integration (for example use a calculus textbook).

Section 2.1: The set \mathbb{R} of real numbers

This section involves a consideration of properties of the set \mathbb{R} of real numbers, the set \mathbb{Q} of rational numbers, the set \mathbb{Z} of integers and other related sets of numbers.

We will be interested in

- what is special about \mathbb{R}
- what distinguishes the real numbers from the rational numbers
- why the set of real numbers is such an interesting and important thing that there is a whole branch of mathematics (real analysis) devoted to its study.

What is \mathbb{R} ?

There are at least two useful ways to think about what real numbers are. First we recall what *integers* are and what *rational numbers* are.

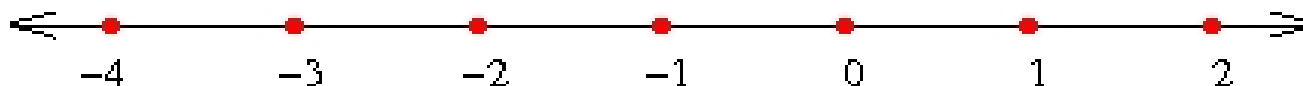
Integers are “whole numbers”. The set of integers is denoted by \mathbb{Z} :

$$\mathbb{Z} = \{ \dots, -2, -1, 0, 1, 2, \dots \}.$$

The notation “ \mathbb{Z} ” comes from the German word *Zahlen* (numbers).

The Integers \mathbb{Z}

On the number line, the integers appear as an infinite set of evenly spaced points. The integers are exactly those numbers whose decimal representations have all zeroes after the decimal point.



Note that the integers on the number line are separated by *gaps*. For example there are no integers in the chunk of the number line between $\frac{7}{5}$ and $\frac{63}{32}$.

Properties of the order on \mathbb{Z}

The set of integers is **well-ordered**.

Given any integer, it makes sense to talk about *the next* integer after that one.

For example, the next integer after 3 is 4.

To see why this property is something worth bothering about, it might be helpful to observe that the same property does not hold for the set \mathbb{Q} of **rational numbers** described below.

The set \mathbb{Q} of rational numbers

A **rational number** is a number that can be expressed as a fraction with an integer as the numerator and a non-zero integer as the denominator.

The set of all rational numbers is denoted by \mathbb{Q} (\mathbb{Q} is for *quotient*).

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}.$$

The rational numbers

So \mathbb{Q} includes such numbers as $\frac{5}{7}$, $-\frac{8}{16}$, $\frac{3141}{22445}$ and so on.

It includes all the integers, since any integer n can be written in the form of a fraction as $\frac{n}{1}$.

The rational numbers are exactly those numbers whose decimal representations either **terminate** (i.e. all digits are 0 from some point onwards) or **repeat**.

(**Challenge**: Prove this statement)

Note: The statement that \mathbb{Q} includes all the integers can be written very concisely as $\mathbb{Z} \subset \mathbb{Q}$.

The geometry of \mathbb{Z} and \mathbb{Q}

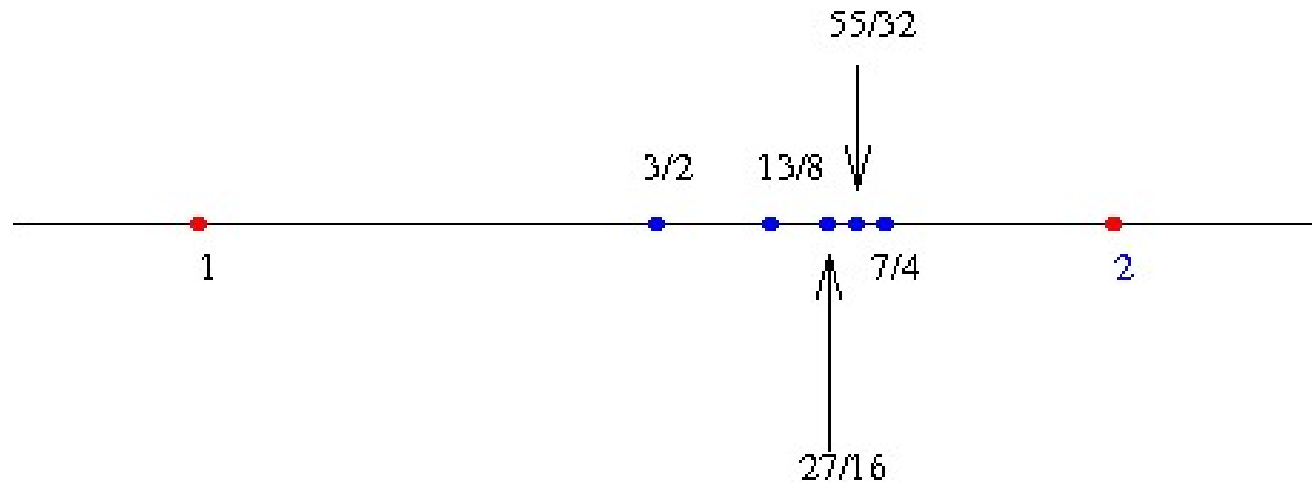
Since rational numbers can be written as quotients (or fractions) involving integers, the sets \mathbb{Q} and \mathbb{Z} are closely related.

However, on the number line these sets do not resemble each other at all. The integers are **spaced out** on the number line and there are gaps between them.

There are no stretches of the number line that are without rational numbers.

The rational numbers are densely packed

The diagram below is intended to show that the stretch of the number line between 1 and 2 contains infinitely many rational numbers.

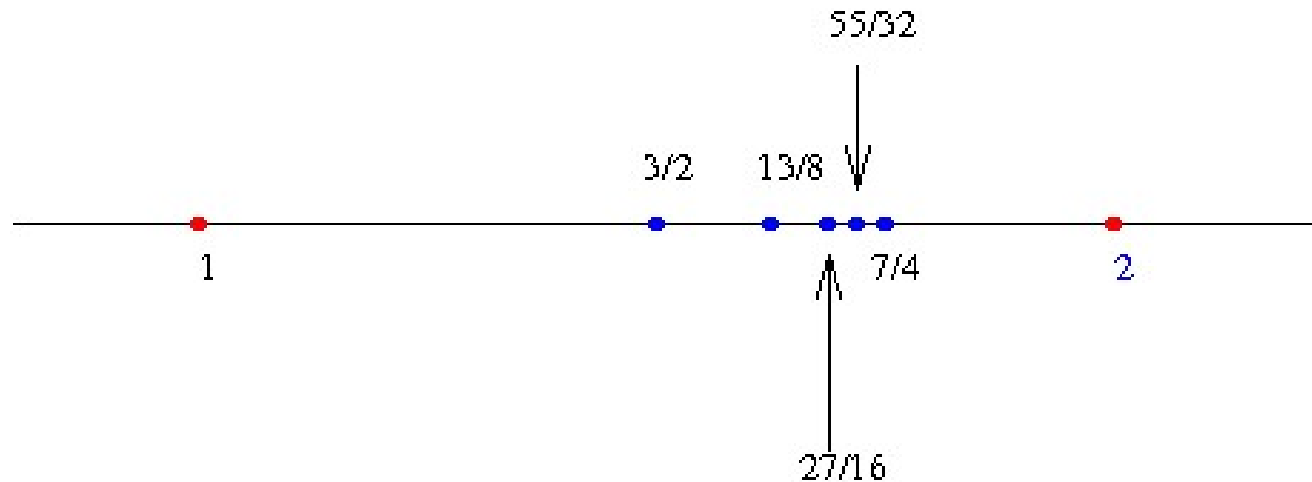


The set of rational numbers is not **well-ordered**.

Given a particular rational number, there is no *next* rational number after it.

The rational numbers are densely packed

The diagram below is intended to show that the stretch of the number line between 1 and 2 contains infinitely many rational numbers.



Exercise 38

Choose a stretch of the number line - for example the stretch from $-\frac{7}{4}$ to $-\frac{11}{8}$ (but pick your own example). Persuade yourself that your stretch contains infinitely many rational numbers.

The real number line

Now imagine an infinite straight line, on which the integers are marked (in order) by an infinite set of evenly spaced dots.

Imagine that the rational numbers have also been marked by dots, so that the dot representing $\frac{3}{2}$ is halfway between the dot representing 1 and the dot representing 2, and so on.

At this stage a lot of dots have been marked - every stretch of the line, no matter how short, contains an infinite number of marked dots. However, many points on the line remain unmarked. For example, somewhere between the dot representing the rational number 1.4142 and the dot representing the rational number 1.4143 is an unmarked point that represents the real number $\sqrt{2}$.

The set \mathbb{R} of *real numbers* is the set of numbers corresponding to *all* points on the line, marked or not.

Not all real numbers are rational

Example

$\sqrt{3}$ is irrational.

Proof (by contradiction). Suppose that $\sqrt{3}$ is rational and write

$$\sqrt{3} = \frac{m}{n},$$

where m and n are positive integers with no common integer factors.

Then

$$3 = \frac{m^2}{n^2} \implies m^2 = 3n^2.$$

This means that m^2 is a multiple of 3, and so m is a multiple of 3, which means that m^2 is actually a multiple of 9; write $m^2 = 9k$, for some $k \in \mathbb{Z}$. Then

$$m^2 = 9k = 3n^2 \implies n^2 = 3k,$$

so n^2 is a multiple of 3, hence so is n . But now both m and n are multiples of 3, which means there is no way to write $\sqrt{3}$ in the form $\frac{m}{n}$ for integers m and n with no common factors.

Rational and Irrational Numbers

Remark: Because the examples of irrational numbers that are usually cited are things like $\sqrt{2}$, π and e , you could get the impression that irrational numbers are special and rare.

This is far from being true!

In a very precise way that we will see later, the irrational numbers are more numerous than the rational numbers.

Exercise 39

Write down five irrational numbers between 4 and 5.

Exercise about rational and irrational numbers

Exercise 40

Suppose that a is a rational number and b is an irrational number.

- *Might $a + b$ be irrational?*
- *Must $a + b$ be irrational?*
- *Might ab be irrational?*
- *Must ab be irrational?*
- *Might the product of two rational numbers be rational?*
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- *Must the product of two irrational numbers be irrational?*

To conclude this section we propose two different ways of thinking about the set of real numbers.

Arithmetic description: The set \mathbb{R} of real numbers consists of *all* numbers that can be written as (possibly non-terminating and possibly non-repeating) decimals.

This description is conceptually valuable but not of much practical use.

All numbers that can be expressed as decimals means *all* numbers that can be written as sequences of the digits $0, 1, \dots, 9$ (with a decimal point somewhere) with no pattern of repetition necessary in the digits. In the universe of all such things, the ones that terminate or have a repeating pattern from some point onwards are **special and rare**. These are the **rational numbers**. The ones that have all zeroes after the decimal point are even more special - these are the **integers**.

To conclude this section we propose two different ways of thinking about the set of real numbers.

Geometric description The set \mathbb{R} of real numbers is the set of *all* points on the number line. This is a *continuum* - there are no gaps in the real numbers and no point on the line that doesn't correspond to a real number.

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Geometric description The set \mathbb{R} of real numbers is the set of *all* points on the number line. This is a *continuum* - there are no gaps in the real numbers and no point on the line that doesn't correspond to a real number.

Note: As this course proceeds you will need to know what the symbols \mathbb{Z} , \mathbb{Q} and \mathbb{R} mean and be able to recall this information easily. You'll need to be familiar with all the notation involving sets etc. that is used in this section and to be able to use it in an accurate way.

Learning outcomes for Section 2.1

After studying this section you should be able to:

- Use the notation \mathbb{Z} , \mathbb{Q} and \mathbb{R} correctly and reliably, and describe the elements of these sets.
- Explain how these sets appear on the number line, and point out some important differences between them.