MA203/MA283: LINEAR ALGEBRA
SEMESTER 2 2024-25
PRACTICE PROBLEM SHEET 2

Definition Let A be a $m \times n$ matrix. The *transpose* of A, denoted A^T, is the $n \times m$ matrix whose entries are given by $(A^T)_{ij} = A_{ji}$. The entries of the first column of A^T are the entries of the first row of A, the entries of the second column of A^T are the entries of the second row of A, etc.

1. Write down the transpose of each of the following matrices.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ 2 & 1 & 0 \\ -3 & 4 & 4 \end{bmatrix}, \begin{bmatrix} 5 & -6 & 2 \\ 4 & 0 & 4 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 4 \\ 2 & 4 & 10 \end{bmatrix}.$$

2. Use elementary row operations to find the inverse of the matrix $\begin{bmatrix} 1 & 4 & -3 \\ -2 & 2 & -4 \\ 3 & 1 & 0 \end{bmatrix}$.

3. (Associativity of Matrix Multiplication) Suppose that A, B, C are matrices of sizes $m \times p$, $p \times q$ and $q \times n$ respectively. By considering the entry in the (i, j) position on each side, show that $(AB)C = A(BC)$.

4. A square matrix is called *orthogonal* if its inverse is equal to its transpose.
   (a) Give two examples of 2×2 orthogonal matrices.
   (b) If A and B are orthogonal matrices of the same size, prove that AB is also orthogonal.

5. (a) If A and B are any matrices for which the product AB is defined, show that $(AB)^T = B^T A^T$. (Hint: Write down how the entry in the (i, j) position of $(AB)^T$ depends on the entries of A and B. If you are not sure what to do, try this out for a particular pair of (small) matrices, verify that it is true and try to figure out why.)
   (b) A square matrix is *symmetric* if it is equal to its own transpose. Verify that for every matrix A, the matrices AA^T and $A^T A$ are sqare and symmetric. (Hint: use part (a). Try with an example if you are not sure what this is about).
   (c) Find all 2×2 matrices that are both symmetric and orthogonal.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformations with

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}.$$

   Write down the matrices of T, S and the compositions $T \circ S$ and $S \circ T$.

7. For the linear transformations T and S of Question 7 above, find the image of $\begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$ under $S \circ T$.

8. For the linear transformation T of Question 7 above, find a non-zero vector $v \in \mathbb{R}^3$ for which $T(v) = \binom{0}{0}$.

9. Suppose that $T : \mathbb{R}^n \to R^p$ and $S : R^p \to \mathbb{R}^m$ are linear transformations. Use the defining properties of linear transformations to deduce that $S \circ T : \mathbb{R}^n \to R^m$ is a linear transformation.

10. Which of the following are subspaces of the vector space $M_2(\mathbb{R})$ consisting of all 2×2 matrices with real entries?
    (a) The set of all invertible matrices.
    (b) The set of all non-invertible matrices.
    (c) The set consisting only of the zero matrix.

(d) The set $M_2(\mathbb{Q})$ of all matrices whose entries are rational.

(e) The set of all matrices whose entry in the $(2,1)$-position is 0.

(f) The set of all matrices whose four entries sum to 0.

(g) The set of all matrices that have at least one entry equal to 0.

(h) The set of all symmetric matrices (A is symmetric if $A = A^T$).

11. Which of the following are spanning sets of $M_2(\mathbb{R})$?

(a) The set of all invertible matrices.

(b) The set of all non-invertible matrices.

(c) The set $M_2(\mathbb{Q})$ of all matrices whose entries are rational.

(d) The set of all matrices whose entry in the $(2,1)$-position is 0.

(e) The set of all matrices whose entry in the $(2,1)$-position is 1.

(f) The set of all matrices whose four entries sum to 0.

(g) The set of all matrices that have at least one entry equal to 0.

12. Determine whether or not each of the following is a spanning set of the set P_4 of all polynomials in $\mathbb{R}[x]$ of degree at most 3.

(a) $\{x^3 + x^2, x^2 + x, x + 1\}$

(b) $\{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1, 1\}$

(c) $\{x^3 + x^2 + x - 1, x^3 + x^2 - x + 1, x^3 - x^2 + x + 1, -x^3 + x^2 + x + 1\}$

(d) $\{x^3 + x^2 + x - 1, x^3 + x^2 - x + 1, x^3 - x^2 + x + 1, x^3 + x^2\}$

13. Prove that $\mathbb{R}[x]$ does not have a finite spanning set as a vector space over $\mathbb{R}$.