Lecture 6: Matrix Algebra

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What is Matrix Algebra?

An algebraic structure is a set of elements (numbers, functions, matrices, polynomials, ...) equipped with some arithmetic operation(s), such as addition, subtraction, multiplication, composition, ...

A matrix is a rectangular array of numbers. So far in this course, matrices have mostly been regarded as tables of coefficients in linear systems. We haven't emphasized their algebraic properties (although we have been secretly factorizing matrices!).

The matrix algebra viewpoint is that matrices are themselves equipped with algebraic operations of addition, matrix multiplication, and multiplication by scalars.

Matrix Addition

If a matrix has m row and n columns, its size is called $m \times n$ (read this as "m by n"). Two matrices can be added together if (and only if) they have the same size. We just add the entries in each position.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1+2 & 3+0 & -1+1 \\ 2+3 & -1+(-1) & 1+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 5 & -2 & 0 \end{bmatrix}$$

The $m \times n$ zero matrix is the $m \times n$ matrix whose entries are all zeros. It is the identity element for addition of $m \times n$ matrices - this means that adding it to another $m \times n$ matrix has no effect.

Multiplying a matrix by a scalar

This means multiplying each of its entries by that scalar. For example

$$3\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 1\end{bmatrix} = \begin{bmatrix}3 & 9 & -6\\ 6 & -3 & 3\end{bmatrix}, \quad -1\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 1\end{bmatrix} = \begin{bmatrix}-1 & -3 & 2\\ -2 & 1 & -1\end{bmatrix}$$

With these operations of addition and scalar multiplication, the set of $m \times n$ matrices is a *vector space*.

A vector space is an algebraic structure whose elements can be added, subtracted and multiplied by scalars, subject to some compatibility conditions (see the lecture notes for the details).

Remark Now that we have addition and scalar multiplication, we can also subtract matrices (define $A - B = A + (-1)B$), provided A and B have the same size), and compute any expression of the form $aB + bB + cC$, for matrices A, B, C of the same size, and any scalars a, b, c.

Linear combinations

Definition

Suppose that $v_1, v_2, \dots, v_k$ are elements that can be added together or multiplied by scalars[1]*. A linear combination of $v_1, \dots, v_k$ is an element of the form*

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k,$$

where the a_i are scalars. In this situation the a_i are called the coefficients in the linear combination.

The term *linear combination* is very intrinsic to the language of linear algebra, we need to understand it well.

Question Which of the following are $\mathbb{R}$-linear combinations of the row vectors $[1\ -2\ 2]$ and $[4\ 0\ 1]$?

(a) $[-1\ -6\ 5]$ (b) $[2\ 4\ 0]$

[1] Examples include vectors in $\mathbb{R}^n$, matrices of the same size, polynomials, functions $\mathbb{R} \to \mathbb{R}$, ...

Matrix Multiplication I: Matrix-Vector Multiplication

We can sometimes also multiply matrices by matrices.

Definition

Let A be a $m \times n$ matrix and let v be a column vector with n entries (a $n \times 1$ matrix). Then the matrix-vector product Av is the column vector obtained by taking the linear combination of the columns of A whose coefficients are the entries of v. It is a column vector with m entries.

Example

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$

Remark Av (if it is defined) has the same size as a single column of A.

Another interpretation of Linear Systems

The linear system $\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$ may be interpreted as the matrix equation

$$\left[\begin{array}{rrr} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 5 \\ 9 \\ 0 \end{array}\right]$$

A solution means an expression for $\left[\begin{array}{c} 5 \\ 9 \\ 0 \end{array}\right]$ as a linear combination of the three columns of the matrix $A = \left[\begin{array}{rrr} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{array}\right]$. The set of all such combinations is called the column space of A.