Lecture 12: Linear Independence

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Linear Dependence and Linear Independence

For a subset $\{v_1, \dots, v_k\}$ of $\mathbb{R}^n$, suppose that v_k is a linear combination of $v_1, \dots, v_{k-1}$. Then every linear combination of $v_1, \dots, v_k$ is "already" a linear combination of $v_1, \dots, v_{k-1}$ and

$$\langle v_1, \dots, v_k \rangle = \langle v_1, \dots, v_{k-1} \rangle.$$

If we are interested in the span of $\{v_1, \dots, v_k\}$ we could throw away v_k and this would not change the span.

Definition A set of (at least two) vectors in $\mathbb{R}^n$ is linearly dependent if one of its elements is a linear combination of the others.
A set of vectors in R^n is linearly independent if it is not linearly dependent.[1]

Linear independence means that throwing away any element results in shrinking the span.

[1] Small print: a set with just one vector is linearly independent, unless this vector is the zero vector. Any set that contains the zero vector is linearly dependent.

More on Linear Independence

Example from Lecture 2

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array} \quad \begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three equations of the system, or the three rows of the original augmented matrix, formed a *linearly dependent set*. One row was eliminated by adding a linear combination of the other two. All the information in the system was contained in just (any) two of the three equations.

The non-zero rows of the reduced row-echelon form are linearly independent, and they span the rowspace of the original matrix. The rowspace is the subspace of $\mathbb{R}^5$ that is spanned by the rows.

Meaning of linear independence - theory and practice

Meaning of linear independence A set is linearly independent if none of its elements is a linear combination of the others.

This definition makes conceptual sense, but to use it as a test for linear independence would mean checking it separately for every element of the set - not so efficient. We have an alternative formulation for this purpose, which is logically equivalent but maybe a bit more obscure as a description.

> A set of vectors is linearly independent if the only way to write the zero vector as a linear combination of its elements is by taking all the coefficients to be zero.

Test for linear independence

To decide if the set $\{v_1, \dots, v_k\}$ is linearly independent, try to write the zero vector as a linear combination of the v_i:

$$\sum_{i=1}^{k} a_i v_i = a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0,$$

for scalars $a_1, \dots, a_k$. If $a_i = 0$ for every i is the only solution, then $v_1, \dots, v_k$ are linearly independent. If there is another solution, they are linearly dependent.

Example Decide whether $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^3$. Solution By row reduction we find

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies a = b = c = 0.$$

Conclusion The set is linearly independent.