Lecture 10: Linear Transformations and Subspaces

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Linear Transformations

Linear transformations are the primary functions between vector spaces that are of interest in linear algebra. They are special because they cooperate with the algebraic structure.

Definition Let m and n be positive integers. A linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies

- $T(u + v) = T(u) + T(v)$, and
- $T(\lambda v) = \lambda T(v)$,

for all u and v in $\mathbb{R}^n$, and all scalars $\lambda \in \mathbb{R}$.

The Matrix of a Linear Transformation

Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, with

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix},\ T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix},\ T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-6\\7\end{bmatrix}$$

Then for the vector in $\mathbb{R}^3$ with any entries a, b, c

$$T\begin{bmatrix}a\\b\\c\end{bmatrix} = aT\begin{bmatrix}1\\0\\0\end{bmatrix} + bT\begin{bmatrix}0\\1\\0\end{bmatrix} + cT\begin{bmatrix}0\\0\\1\end{bmatrix} = \underbrace{\begin{bmatrix}2 & 1 & -6\\-3 & 4 & 7\end{bmatrix}}_{M_T}\begin{bmatrix}a\\b\\c\end{bmatrix}.$$

and the 2×3 matrix M_T is called the (standard) matrix of T.

The matrix of a linear transformation

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by a $m \times n$ matrix M_T. The columns of M_T are the images under T of the standard basis vectors $e_1, \dots, e_n$.
- If v is any vector in $\mathbb{R}^n$, we can calculate $T(v)$ by multiplying the column vector v on the left by the matrix M_T. Matrix-vector multiplication is evaluating linear transformations.
- On the other hand, if A is any $m \times n$ matrix, then A determines a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ by $v \to Av$, for $v \in \mathbb{R}^n$. So, in a sense, matrices are linear transformations.
- Examples of linear transformations include rotations, reflections and scaling, but not translations.
- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then in order to evaluate T at any point/vector, we only need mn pieces of information, just the m coordinates of each of the n images of the standard basis vectors. This is very different for example from continuous functions from $\mathbb{R}$ to $\mathbb{R}$ - we cannot know all about them just by knowing their values at a few points.

Matrix multiplication is composition

Suppose that $T : \mathbb{R}^n \to \mathbb{R}^p$ and $S : \mathbb{R}^p \to \mathbb{R}^m$ are linear transformations. Then $S \circ T$ (S after T) is the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined for $v \in \mathbb{R}^n$ by

$$S \circ T(v) = S(T(v)).$$

Question How does the $(m \times n)$ matrix $M_{S \circ T}$ of $S \circ T$ depend on the $(m \times p)$ matrix M_S of S and the $(p \times n)$ matrix M_T of T?

To answer this we have to think about the definition of $M_{S \circ T}$.

- Its first column has the coordinates of $S \circ T(e_1) = S(T(e_1))$.
- $T(e_1)$ is the first column of M_T.
- Then $S(T(e_1))$ is the matrix-vector product M_S[first column of M_T]. This is the first column of the matrix product $M_S M_T$.
- Same for all the other columns: the conclusion is $M_{S \circ T} = M_S M_T$.

Matrix multiplication is composition of linear transformations.

Corollary Matrix multiplication is associative.

The Image and Kernel of a Linear Transformation

$T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation with $M_T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix}$.

The image of T is the subset of $\mathbb{R}^3$ consisting of all elements $T(v)$, where $v \in \mathbb{R}^3$. This is the set of all vectors of the form

$$a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}.$$

In matrix terms, this is the column space of M_T.

The kernel of T is the set of all vectors v in $\mathbb{R}^3$ with $T(v) = 0$. This is the set of all column vectors whose entries a, b, c satisfy

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In matrix terms this is the (right) nullspace of M_T.

Example: The kernel is a **line** and the image is a **plane**

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & -1 & 5 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The kernel/nullspace is $\boxed{\{(-2,1,1)t,\ \ t \in \mathbb{R}\}}$ a line in $\mathbb{R}^3$.

That $(-2,1,1)$ is in the kernel of T means that (for example) Column 3 of M_T is a linear combination of Columns 1 and 2.

$$-2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

It follows that every linear combination of all three columns of M_T is actually a linear combination just of Columns 1 and 2.

The column space of M_T is $\left\{ a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} :\ a, b \in \mathbb{R} \right\}$, a plane in $\mathbb{R}^3$.

Subspaces

Definition A (non-empty) subset V of $\mathbb{R}^n$ is a subspace if

- It is closed under addition: $u + v \in V$ whenever $u \in V$ and $v \in V$.
- It is closed under scalar multiplication: $ku \in V$ whenever $u \in V$ and $k \in \mathbb{R}$.

Examples

1. $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$ is not a subspace of $\mathbb{R}^3$. The vectors $(1, 0, 0)$ and $(0, 1, 0)$ belong to this set but their sum $(1, 1, 0)$ does not.
2. $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) = 0\}$ is a subspace of $\mathbb{R}^3$.
3. $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) \neq 0\}$ is not a subspace of $\mathbb{R}^3$. For example, $(1, 4, 1)$ and $(-5, -2, -1)$ belong to this set but their sum $(-4, 2, 0)$ does not.
4. The kernel of any linear transformation is a subspace.
5. The image of any linear transformation is a subspace.

Exercise Prove these last two points.