To calculate

$$\int_a^b f(x)dx$$

1. Find a function F for which $F'(x) = f(x)$, i.e. find a function F whose derivative is f.
2. Evaluate F at the limits of integration a and b; i.e. calcuate $F(a)$ and $F(b)$. This means replacing x separately with a and b in the formula that defines $F(x)$.
3. Calculate the number $F(b) - F(a)$. This is the definite integral $\displaystyle\int_a^b f(x)\,dx$.

Of the three steps above, the first one is the hard one.

Notation

Recall the following notation : if F is a function that satisfies $F'(x) = f(x)$, then

$$F(x)\big|_a^b \text{ or } F(x)\big|_{x=a}^{x=b} \text{ means } F(b) - F(a).$$

Definition 14

Let f be a function. Another function F is called an antiderivative of f if the derivative of F is f, i.e. if $F'(x) = f(x)$, for all (relevant) values of the variable x.

So for example x^2 is an antiderivative of $2x$. Note that $x^2 + 1$, $x^2 + 5$ and $x^2 - 20e$ are also antiderivatives of $2x$. So we talk about an antiderviative of a function or expression rather that the antiderivative.

The Indefinite Integral

Definition 15

Let f be a function. The indefinite integral of f, written

$$\int f(x)\,dx$$

is the "general antiderivative" of f. If $F(x)$ is a particular antiderivative of f, then we would write

$$\int f(x)\,dx = F(x) + C,$$

to indicate that the different antiderivatives of f look like $F(x) + C$, where C may be any constant. (In this context C is often referred to as a constant of integration).

Examples

Example 16

Determine $\int \cos 2x \, dx$.

Solution: The question is: what do we need to differentiate to get $\cos 2x$? Well, what do we need to differentiate to get something involving $\cos$?

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Examples

Example 16

Determine $\int \cos 2x \, dx$.

Solution: The question is: what do we need to differentiate to get $\cos 2x$? Well, what do we need to differentiate to get something involving cos? The derivative of $\sin x$ is $\cos x$. A reasonable guess would say that the derivative of $\sin 2x$ might be "something like" $\cos 2x$.
By the chain rule, the derivative of $\sin 2x$ is in fact $2 \cos 2x$.
So $\sin 2x$ is pretty close but it gives us twice what we want - we should compensate for this by taking $\frac{1}{2} \sin 2x$; its derivative is

$$\frac{1}{2}(2 \cos 2x) = \cos 2x.$$

Conclusion: $\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$

Powers of x

Example 17

Determine $\int x^n \, dx$

Important Note: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an antiderivative of x^n we have to reverse this process. This means that the index increases by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

The Integral of $\frac{1}{x}$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x}\,dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

This latter formula applies for all $x \neq 0$.

A definite integral

Example 18

Determine $\int_0^{\pi} \sin x + \cos x \, dx.$

Solution: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned} \int_0^{\pi} \sin x + \cos x \, dx &= -\cos x + \sin x \Big|_0^{\pi} \\ &= (-\cos \pi + \sin \pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2. \end{aligned}$$

A definite integral

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Note: To determine $\cos \pi$, start at the point $(1, 0)$ and travel counter–clockwise along the the unit circle for a distance of π, arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos \pi$, and the y-coordinate is $\sin \pi$.

1.4.1 Substitution - Reversing the Chain Rule

The Chain Rule of Differentation tells us that in order to differentiate the expression $\sin x^2$, we should regard this expression as sin("something") whose derivative (with respect to "something") is cos("something"), then multiply this expression by the derivative of the "something" with respect to x. Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2.$$

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$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2.$$

Equivalently

$$\int 2x \cos x^2 \, dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from $2x \cos x^2$ back to $\sin x^2$.

How Substitution Works

Example 19

Determine $\int 2x\sqrt{x^2+1}\,dx$.

Solution Notice that the integrand involves both the expressions x^2+1 and $2x$. Note also that $2x$ is the derivative of x^2+1.

1. Introduce the notation u and set $u = x^2+1$.
2. Note $\dfrac{du}{dx} = 2x$; rewrite this as $du = 2x\,dx$.
3. Then
$$\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}(2x\,dx) = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C.$$
4. So
$$\int 2x\sqrt{x^2+1}\,dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

Substitution and definite integrals

Example 20

Determine $\int_0^{\pi} \cos^3 x \sin x \, dx$ *(from 2015 Summer paper)*

Solution: Write $u = \cos x$. Then

$$\frac{du}{dx} = -\sin x, \;\; du = -\sin x \, dx, \quad \sin x \, dx = -du.$$

Change variables: $\int_0^{\pi} \cos^3 x \, \sin x \, dx = -\int_{x=0}^{x=\pi} u^3 \, du$. Limits of integration: When $x = 0$, $u = \cos x = \cos 0 = 1$. When $x = \pi$, $u = \cos x = \cos \pi = -1$. Our integral becomes:

$$\int_{u=1}^{u=-1} u^3 \, du = \left.\frac{u^4}{4}\right|_{u=-1}^{u=1} = \frac{1}{4} - \frac{(-1)^4}{4} = 0.$$

Substitution and Definite Integrals - more examples

Example 21

Evaluate $\displaystyle\int_0^1 \frac{5r}{(4+r^2)^2}\,dr.$

Solution To find an antiderivative, let $u = 4 + r^2$.

Then $\dfrac{du}{dr} = 2r,\ du = 2r\,dr;\ 5r\,dr = \dfrac{5}{2}du.$

So

$$\int \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int \frac{1}{u^2}\,du = \frac{5}{2}\int u^{-2}\,du.$$

Thus

$$\int \frac{5r}{(4+r^2)^2}\,dr = -\frac{5}{2}\times\frac{1}{u} + C,$$

and we need to evaluate $-\frac{5}{2}\times\frac{1}{u}$ at $r = 0$ and at $r = 1$. We have two choices.

Two Choices

1. Write $u = 4 + r^2$ to obtain

$$\begin{aligned}
\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2} \times \frac{1}{4+r^2}\bigg|_{r=0}^{r=1} \\
&= -\frac{5}{2} \times \frac{1}{4+1^2} - \left(-\frac{5}{2} \times \frac{1}{4+0^2}\right) \\
&= -\frac{5}{2} \times \frac{1}{5} + \frac{5}{2} \times \frac{1}{4} \\
&= \frac{1}{8}.
\end{aligned}$$

2. Alternatively, write the antiderivative as $-\frac{5}{2} \times \frac{1}{u}$ and replace the limits of integration with the corresponding values of u.
   When $r = 0$ we have $u = 4 + 0^2 = 4$.
   When $r = 1$ we have $u = 4 + 1^2 = 5$.
   Thus

$$\begin{aligned}
\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2} \times \frac{1}{u}\Big|_{u=4}^{u=5} \\
&= -\frac{5}{2} \times \frac{1}{5} - \left(-\frac{5}{2} \times \frac{1}{4}\right) \\
&= \frac{1}{8}.
\end{aligned}$$

From Summer Exam 2013

Example 22

Determine

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx.$$

Solution: Write

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx.$$

Now write $u=\sqrt{x}+1$. Then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} \Longrightarrow \frac{1}{\sqrt{x}}\,dx = 2du.$

Then

$$\begin{aligned} \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx &= \int_{x=1}^{x=4} \frac{2}{u}\,du = \int_{u=2}^{u=3} \frac{2}{u}\,du = 2\ln u\big|_2^3 \\ &= 2(\ln 3 - \ln 2) = 2\ln\frac{3}{2}. \end{aligned}$$

Example 23

Determine $\int (1 - \cos t)^2 \sin t \, dt$

More Examples

Example 23

Determine $\int (1 - \cos t)^2 \sin t \, dt$

Question: *How do we know what expression to extract and refer to as u?* Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.

More Examples

Example 23

Determine $\int (1 - \cos t)^2 \sin t \, dt$

Question: *How do we know what expression to extract and refer to as u?* Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.
There is no easy answer but with practice we can develop a sense of what might work. In this example the integrand involves the expression $1 - \cos t$ and also its derivative $\sin t$. This is what makes the substitution $u = 1 - \cos t$ effective for this problem.

NOTE: There are more examples of the substitution technique in the lecture notes.