Lecture 4: Visualizing the solution to a linear system

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Lecture 4: Visualizing solutions of linear systems

Euclidean space

Definition A column vector (with n entries) is a $n \times 1$ matrix (a matrix with n entries arranged in one column). A row vector is a matrix with one row.

Definition Euclidean n-dimensional space $\mathbb{R}^n$ is the set of all (row or column)[1] vectors with n entries.

Examples

$[1\ 4\ \ -2]$ is a row vector in $\mathbb{R}^3$.
$[0,\ \pi,\ -\sqrt{2},\ 1,\ 20]$ is a row vector in $\mathbb{R}^5$ (commas are optional).

$$\begin{bmatrix} 3 \\ -4 \\ -2 \\ 5.4 \end{bmatrix}$$ is a column vector in $\mathbb{R}^4$.

Remark $\mathbb{R}^2$ and $\mathbb{R}^3$ are equipped with orthogonal coordinate axes that we know well. It is useful to imagine $\mathbb{R}^n$ as having n orthogonal coordinate axes, even though we can't fit them in our physical environment (more later on that).

[1] we can decide on any occasion if we mean row or column vectors, that will be ok as long as we are consistent within the discussion

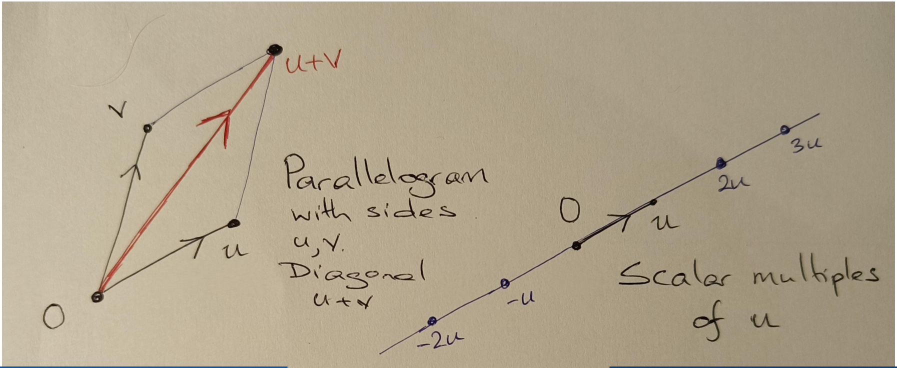
Addition and Scalar Multiplication in $\mathbb{R}^n$

If $u, v \in \mathbb{R}^n$, we define $u + v$ to be the vector that we get by adding the coordinates of u and v. For example in $\mathbb{R}^3$

$$[1\ 5\ \ -2] + [-2\ 3\ 2] = [1 + (-2)\ 5 + 3\ \ -2 + 2] = [-1\ 8\ 0].$$

We can multiply a vector by a scalar (number), just multiply all the coordinates: $4[1\ 3\ \ -1] = [4\ 12\ \ -4]$.

Geometrically:



Solution set of a system with two free variables

Recall from Lecture 2 The system

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array}$$

has general solution

$$(x_1, x_2, x_3, x_4) = (2, 1, 0, 0) + s(1, -2, 1, 0) + t(-3, 4, 0, 1), \ \ s, t \in \mathbb{R}.$$

What does this look like as a subset of $\mathbb{R}^4$?

It consists all points that we can get by adding a scalar multiple of $u = (1, -2, 1, 0)$ and a scalar multiple of $v = (-3, 4, 0, 1)$ to $(2, 1, 0, 0)$. Just $(2, 1, 0, 0) + s(1, -2, 1, 0)$ would give us the line through $(2, 1, 0, 0)$

parallel to the vector $[1 \ \ -2 \ 1 \ 0]$.

Allowing the addition of any multiple of $[-3, 4, 0, 1]$ to any point on this line gives us a plane; it looks like a copy of $\mathbb{R}^2$ inside $\mathbb{R}^4$.

Putting Descartes before the horse . . .

Coordinate geometry was invented by René Descartes in the first half of the 17th Century (possibly independently by other people). It allows us to interpret a (row or column) vector as either

- the point whose coordinates are the entries of the vector, or
- the line segment directed from the origin to that point.

The coordinate setup means that any equation involving variables x, y, z or $x_1, x_2, \dots, x_n$ (like $x^2 + y^3 - 2z = 0$, or the linear equation $2x_1 - 3x_2 + 4x_3 + x_4 = 2$) can be interpreted as a geometric object in the relevant $\mathbb{R}^n$, consisting of all those points whose coordinates satisfy the equation.

Figuring out what this looks like is generally difficult, but for linear equations it's ok. Finding the simultaneous solutions of a bunch of equations means finding the intersection of the corresponding geometric objects.