Lecture 3: Inconsistent Systems (and a computational tool)

January 21, 2025

Another Example

Example Consider the following system.

$$\begin{array}{rcrcrcl} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \longleftrightarrow \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array}\right]$$

$$\quad x \quad y \quad z$$

To find solutions, apply elementary row operations to the augmented matrix, with the goal of reducing it (fully or partly[1]) to reduced row echelon form (RREF). The process of reducing to RREF is called Gauss-Jordan elimination.

1. Add a multiple of one row to another
2. Swap two rows
3. Multiply a row by a non-zero scalar

$\longrightarrow$ RREF

[1] partly means maybe stop early if you can see the outcome

Implementation of Gauss-Jordan elimination on a matrix M

1. Get a 1 in the upper left entry (unless Column 1 is all zero).
2. Use this first leading 1 to "clear out" the rest of Column 1, by adding/subtracting multiples of Row 1 to/from subsequent rows.
   DO NOT TOUCH ROW 1 AGAIN UNTIL STEP 5.
3. Move to Column 2. Without disturbing the pattern of zeros in Column 1, get a 1 as the second entry of Row 2 (unless Column 2 has all zeros after the first entry). Use this second leading 1 to clear out Column 2 below the second entry.
   DO NOT TOUCH ROW 2 AGAIN UNTIL STEP 5.
4. Proceed through the columns in this way, clearing them out below the leading 1's. This gives a row echelon form (REF).
5. Finally, use the leading 1s to clear out the columns above them, by adding/subtracting lower rows from higher ones, working from left to right.

Remark Different REFs can be obtained from the same starting point, by different choices of EROs, but the RREF is unique.

Exercise think about how to prove this.

Useful Software - Octave Online

The instructions on the last slide (more or less) constitute an algorithm for Gauss-Jordan elimination.
Many computer algebra systems have built-in functionality for row reduction.

Go to `https://octave-online.net` and just use the interface there.
Or set up a free account if you want to save files.
Or upgrade to a not quite free account if the ads are annoying you.

Syntax to enter our augmented matrix at the prompt $\rangle$:

```
A=[3,2,-5,4;1,1,-2,1;5,3,-8,6]
```

and to calculate its RREF:

```
rref(A)
```

Back to the example

$$\begin{array}{rcrcrcl} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \left[\begin{array}{ccc|c} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{array}\right] \\ \begin{array}{ccc} x & y & \quad z \end{array} \end{array}$$

During the Gaussian elimination process, a row is encountered that has zero entries in all positions except the last. This corresponds to an statement of the form

$$0x + 0y + 0z \neq 0$$

This cannot be satisfied by any values of x, y, z.
Conclusion The system is inconsistent. It has no solution.

A system is inconsistent if its equations cannot be simultaneously satisfied. For example, $x + y = 1,\ x + y = 2$ is inconsistent because $x + y$ can't be simultaneously equal to 1 and 2. Incompatibilities like this are not always immediately noticeable, their visibility emerges during the Gaussian elimination process.

Possible outcomes to solving a linear system

1. The system can be inconsistent. In this case a row with all entries zero except the last occurs during the row reduction. As soon as such a row occurs, STOP and report that there is no solution.
2. If the system is consistent, every variable may be a leading variable. In this case, there is a unique solution.
   Note This can happen only if the number of equations is at least equal to the number of variables, since we need enough rows to accommodate a leading 1 for every variable. A system with four variables and only three equations cannot have a unique solution.
3. The system might be consistent, with one or more free variables. In this case, there are infinitely many solutions, corresponding to different choices of values for the free variables.

Observation it is not possible for a linear system to have exactly two distinct solutions. If it has more than one, it has infinitely many. This was not obvious at the start.

Pitfalls of Gaussian elimination

Algorithms for solving systems of linear equations (and their implementation) continue to be an important field of reserach in computational mathematics. Most methods are based on Gauss-Jordan elimination, with adaptations to manage pitfalls such as the following.

- Numerical instability In computer implementations, matrix entries are rounded or truncated, which introduces "small" errors. These can be amplified during the Gaussian elimination process, to the extent that the computation becomes meaningless. This can occur for example if dividing by a number that is close to zero, which can convert a small inaccuracy to a very large one, that is further propagated by adding and subtracting to other entries.
- Computational complexity The number of arithmetic steps (addition, subtraction, multiplication, division) involved in reducing a $n \times n$ matrix to REF (or RREF) is bounded above by a cubic expression in n. This becomes unfeasible for very large systems, where iterative methods are used.

(a) Find the general solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \end{array}$$

(b) Find the unique value of k for which the following system is consistent.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \\ x_1 & + & x_2 & - & x_3 & - & 5x_4 & = & k \end{array}$$

(c) Find the unique solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 9 \end{array}$$

(a) Find the general solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \end{array}$$

$$\left[\begin{array}{rrrrr} 1 & 3 & 2 & 3 & 6 \\ 2 & -1 & 1 & 8 & -1 \\ 2 & 2 & 3 & 10 & 6 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

The variable x_4 is free. We write t for its value in a solution of the system. Then the general solution is

$$(x_1, x_2, x_3, x_4) = (-1 - t, 1 + 2t, 2 - 4t, t), \ \ t \in \mathbb{R}.$$

(b) Find the unique value of k for which the following system is consistent.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \\ x_1 & + & x_2 & - & x_3 & - & 5x_4 & = & k \end{array}$$

The first three equations here are the same as in part (a). This means that if the system is consistent, any solution has the form

$$(x_1, x_2, x_3, x_4) = (-1 - t, 1 + 2t, 2 - 4t, t),$$

for some value of t.

Writing the fourth equation in terms of t gives

$$(-1 - t) + (1 + 2t) - (2 - 4t) - (5t) = k \Longrightarrow -2 + 0t = k$$

This can be satisfied only if $k = -2$, which answers the question.

Question What are the solutions in this case?

(c) Find the unique solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 9 \end{array}$$

This is like part (b) but describes the more "typical" situation. Again the first three equations come from part (a). Rewriting the fourth equation with the general form of a solution to the first three gives:

$$(-1-t) - 2(1+2t) + (2-4t) - (t) = 9 \implies -1 - 10t = 9, \;\; t = -1$$

The fourth equation adds an extra constraint, just one of the simultaneous solutions to the first three equations also satisfies the fourth. This is the one with $t = -1$:

$$(x_1, x_2, x_3, x_4) = (-1-(-1), 1+2(-1), 2-4(-1), -1) = (0, -1, 6, -1).$$