Euclidean and non-Euclidean Geometry (MA3101)
Lecture 18: Desargues's Theorem

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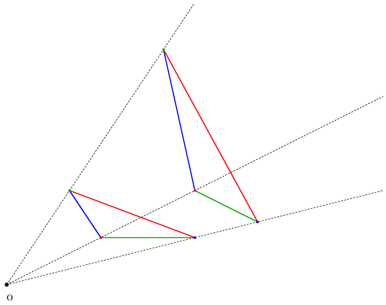
Triangles in Central Perspective

Let ABC and $A'B'C'$ be a pair of triangles in 3-dimensional Euclidean space, with no vertex in common.

Suppose that the lines AA', BB' and CC' all intersect at a point O.

Then we say that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are centrally in perspective from O.



This means that the two triangles would look exactly the same to a viewer at O (in an environment without other reference points).

Two triangles are centrally in perspective if there is some point O that is the intersection point of three distinct lines, each containing one vertex of each triangle.

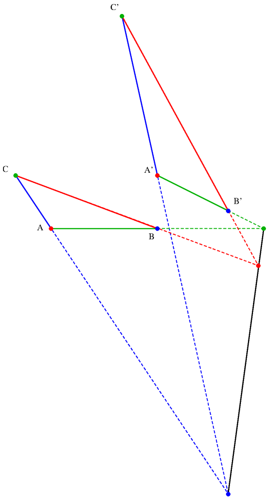
Triangles in Axial Perspective

Let ABC and $A'B'C'$ be a pair of triangles in 3-dimensional Euclidean space, with no vertex in common.

A, A' are coloured red, B, B' are blue, C, C' are green.

Suppose that the lines AB and $A'B'$ (green) intersect in a point, the same for AC and $A'C'$ (blue) and for BC and $B'C'$ (red). Suppose these three intersection points all lie on the same line L.

Then we say that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are axially in perspective from L.

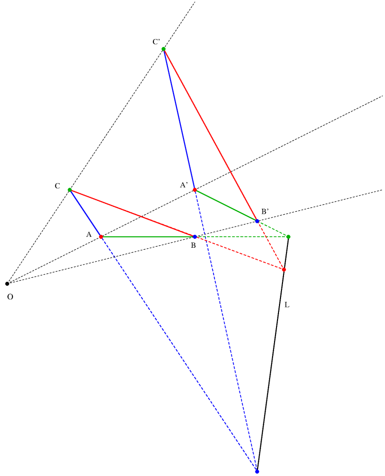


If the edges of one triangle can be matched with the edges of the other, so that the matched pairs of (extended) edges all intersect on a line L, then the two triangles are axially in perspective from L.

Here is the combined version of the first two pictures.

Desargues's Theorem
If two triangles are centrally in perspective, then they are axially in perspective*.

*possibly from a "line at infinity".



Proof of Desargues's Theorem

Let Π and Π' respectively denote the planes in $\mathbb{R}^3$ that contain the triangles ABC and $A'B'C'$. Either $\Pi = \Pi'$ or not: first suppose not (this is the easier case). We assume that the planes Π and Π' are not parallel, so that they intersect in a (Euclidean) line L.

1. The lines AB and $A'B'$ both lie in the plane OAB (or $OA'B'$). If these are not parallel, they intersect at a point in this plane. Since the lines AB and $A'B'$ respectively belong to Π and Π', their point of intersection belongs to $\Pi \cap \Pi' = L$.
2. The same argument applies to the lines AC and $A'C'$: their point of intersection (provided that they are not parallel in the plane OAC) is on L.
3. And the same argument applies to BC and $B'C'$: they intersect at a point of L (provided that they are not parallel in the plane OBC).

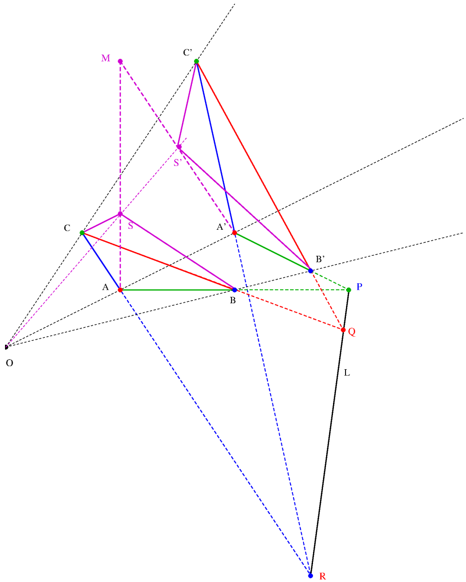
So the two triangles are axially in perspective from L (at least in the "non-parallel" cases).

The "Plane" case ($\Pi = \Pi'$)

Now suppose that the triangles ABC and $A'B'C'$ lie in the same plane (the plane of the picture two slides back). We consider this as a limiting case of the first scenario.

1. We "lift" A and A' out of the plane Π as follows. Fix a point M in 3-dimensional space that does not belong to Π, and consider the line segments MA and MA'.
2. The point O belongs to the plane MAA'. Let S be an interior point of the line segment AM, and let S' be the point where the line OS intersects MA'.
3. The first part of the proof applies to the triangles SBC and $S'BC$: they are axially in perspective along a line L' that includes the point of intersection of BC and $B'C'$.
4. A version of this configuration exists for every choice of the point S: if we slide S towards A along the line MA, then S' slides to B and the line L' approaches a line L in the plane Π, that contains the intersection points of AB and $A'B'$, of AC and $A'C'$, and of BC and $B'C'$.

Picture for the Plane Case



The special "parallel" cases

The theorem extends to include the cases where the planes Π and Π' are parallel, and where a pair of corresponding edges of two triangles are parallel, if we adjust the context to projective space.

- If the planes Π and Π' are parallel, we consider them as representing two copies of $\mathbb{RP}^2$ (inside $\mathbb{RP}^3$) that intersect in a projective line.
- Suppose the planes Π and Π' are different, but the lines AB and $A'B'$ are parallel, so they do not intersect in the (Euclidean) plane OAB.
  Then AB and $A'B'$ represent projective lines that intersect in projective space at a point that belongs to the intersection of Π and Π', interpreted as a projective line.