Euclidean and non-Euclidean Geometry (MA3101)
Lecture 17: Perspectivities

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Incidence of points and lines in $\mathbb{RP}^2$

In $\mathbb{RP}^2$, every pair of distinct points belongs together to a unique line.

This property is shared by the Euclidean plane and the hyperbolic plane, but not the sphere.

Distinct projective points $[a_1 : b_1 : c_1]$ and $[a_2 : b_2 : c_2]$ determine a unique plane Π in $\mathbb{R}^3$ that contains the points $O, P(a_1, b_1, c_1), Q(a_2, b_2, c_2)$. Then $\mathcal{L}\Pi$ is the unique projective line containing $[a_1 : b_1 : c_1]$ and $[a_2 : b_2 : c_2]$ in $\mathbb{RP}^2$.

Every pair of distinct lines in $\mathbb{RP}^2$ intersect in a unique point.

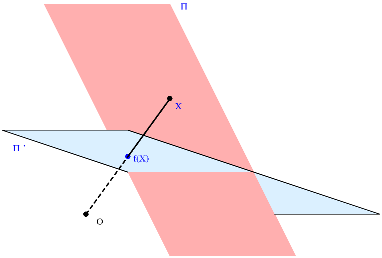
This property is not shared by the Euclidean plane, $\mathcal{H}^2$ or the sphere S^2.
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be distinct projective lines.
Then there are planes Π_1 and Π_2 through O in $\mathbb{R}^3$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively the sets of lines through O in Π_1 and Π_2.
As planes in $\mathbb{R}^3$, Π_1 and Π_2 intersect in a single line through O: this is the point of intersection in $\mathbb{RP}^2$ of the projective lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

Back to window-taping

Suppose that Π and Π' are two planes in $\mathbb{R}^3$ (not passing through the origin O, and not parallel to each other). We define a mapping

$$f : \Pi \to \Pi'$$

by $X \to X'$, where X' is the point of intersection of Π' with the line OX through O and X (and X is any point of Π).



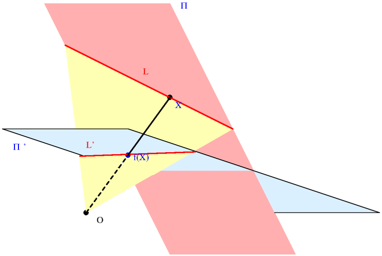
This determines $f(X)$ provided that the line OX is not parallel to Π'.

The mapping $f : \Pi \to \Pi'$ is called a perspectivity.

If Π' is a window and Π shows some scene, the function f translates the scene to what an artist positioned at O would draw, if they faithfully trace what they see on the window.

Perspectivity as a function from one plane to another

For a particular pair of planes Π and Π', we can write down a formulaic description of f. For example if $X = (3, 1, 2)$ belongs to Π, then $f(X)$ is the unique $(3t, t, 2t)$ that satisfies the equation of Π'.



If L is a line in Π, then $f(L)$ is a line in Π'.
The set of lines connecting O to points of L forms a plane (yellow in the picture) that intersects Π' in a line L'. $\boxed{f(L) = L'}$
The exception is when L is K, the intersection of Π with the plane through O parallel to Π'. The domain of f is $\Pi \backslash K$.

The image of f is $\Pi' \backslash K'$, where K' is the intersection of Π' with the plane through O parallel to Π.

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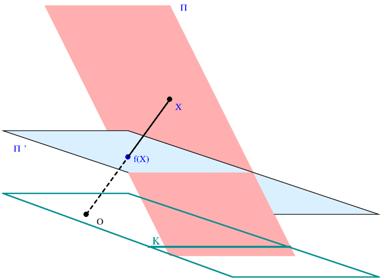

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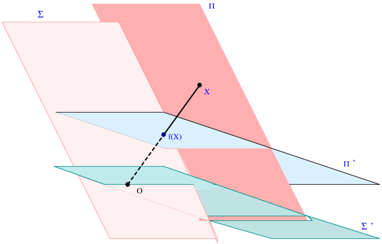
Perspectivity as a function from $\mathbb{RP}^2$ to $\mathbb{RP}^2$

Let Σ and Σ' be the planes through O, respectively parallel to Π and Π'.

Points of $\mathbb{RP}^2$ are lines through O. Each of these intersects Π at one point, except the ones that lie in Σ.

Points of $\mathbb{RP}^2$ can be identified with

- points of Π, and
- lines in Σ through O



or equally well with points of Π' and lines in Σ' through O).
As a mapping from $\mathbb{RP}^2$ to $\mathbb{RP}^2$, f is a bijection, its domain and range are all of $\mathbb{RP}^2$. Recall K is the line $\Pi \cap \Sigma'$.

- For $X \in \Pi \backslash K$, OX is a line through a point of Π'. All ponts of Π' occur this way, except those of $K' = \Pi' \cap \Sigma$
- For $X \in K$, OX is a line through O in Σ'.
- If OX is a line in Σ, then OX corresponds to a point of $\Pi' \cap \Sigma = K'$.