Euclidean and non-Euclidean Geometry (MA3101)
Lecture 16: The Real Projective Plane $\mathbb{RP}^2$

Dr Rachel Quinlan

November 13, 2024

Example from Lecture 15

Example A consistent and an inconsistent system:

$$(A)\ \begin{array}{ccccc} x & + & y & = & 2 \\ x & + & 2y & = & 3 \end{array} \qquad (B)\ \begin{array}{ccccc} x & + & y & = & 2 \\ x & + & y & = & 3 \end{array}$$

Solution sets are $x = y = 1$ and the empty set.
We introduce a new variable z and consider instead the homogeneous systems

$$(A')\ \begin{array}{ccccc} x & + & y & = & 2z \\ x & + & 2y & = & 3z \end{array} \qquad (B')\ \begin{array}{ccccc} x & + & y & = & 2z \\ x & + & y & = & 3z \end{array}$$

Now the first has solution set $\{(t, t, t) : t \in \mathbb{R}\}$.
The second has solution set $\{(t, -t, 0) : t \in \mathbb{R}\}$.

Each of these is a line through the origin in $\mathbb{R}^3$, reflecting the fact that the coefficient matrices have nullity 1.

The Real Projective Plane

If $z' \neq 0$, then (x', y', z') is a solution of (A') if and only if $(\frac{x'}{z'}, \frac{y'}{z'})$ is a solution of (A).

In our example, (x', y', z') is a solution provided $\frac{x'}{z'} = \frac{x'}{y'} = 1$, or

$$[x' : y' : z'] = [1, 1, 1].$$

Same for (B') and (B), but (B') has no solution with $z \neq 0$.

Solutions with $z = 0$: $x = y = z = 0$ for (A'), and $\{(t, -t, 0)\}$ for (B').

How do we interpret these?

Definition We define an equivalence relation on the set of nonzero vectors in $\mathbb{R}^3$ by

$$(a, b, c) \sim (a', b', c') \iff (a', b', c') = \lambda(a, b, c), \lambda \neq 0, \lambda \in \mathbb{R}.$$

Equivalence classes are lines through the origin in $\mathbb{R}^3$.
These classes are the points of the real projective plane $\mathbb{RP}^2$
The class of (a, b, c) is denoted $[a : b : c]$.

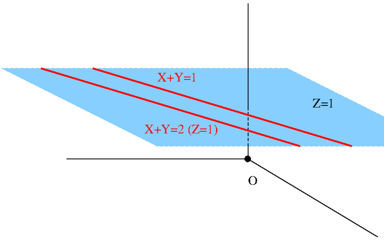
Interpretation of solution of (B') in $\mathbb{RP}^2$

The system (B) is (B') with $Z = 1$.

The plane $Z = 1$ sits above the origin O, parallel to the XY-plane.

Within $Z = 1$, every point Q of the line $X + Y = 1$ determines a unique point of $\mathbb{RP}^2$, namely the line OQ. Same for $X + Y = 2$.

We can slide the point Q along the line $X + Y = 1$ (or $X + Y = 2$) and observe how the line OQ moves. In the distant limit, OQ approaches the line $X + Y = 0$ in the XY-plane.



In $\mathbb{RP}^2$, this Euclidean line is the point $[1 : -1 : 0]$ (or any $[t : -t : 0]$ with $t \neq 0$).

It is an "ordinary" point of the projective plane $\mathbb{RP}^2$, appearing as a "point at infinity" in this particular setup with respect to these parallel lines in the plane $Z = 1$.

Points and Lines in $\mathbb{RP}^2$

A point in $\mathbb{RP}^2$ is a line through O in $\mathbb{R}^3$.

It has homogeneous coordinates $[a : b : c]$, where (a, b, c) is any point of the line other than O. So $[2 : -1 : 2] = [1 : -\frac{1}{2} : 1] = [6 : -3 : 6]$, etc.

A line in $\mathbb{RP}^2$ is the set of lines through O in a $\mathbb{R}^3$ plane Π through O.

How to think of this projective line as a line in any "usual" way?
Take a (Euclidean) line L in Π that does not include O.

Every line through O in Π intersects L in a single point, except the one line L' through O in Π that is parallel to L.

All points of the projective line determined by Π, except for L', are represented by the "ordinary" points of L.

A projective line resembles a Euclidean line with a "point at infinity" added.

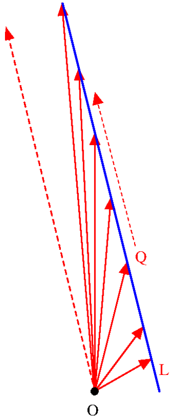
Every point is a "point at infinity"

No point of $\mathbb{RP}^2$ is intrinsically a "point at infinity". Any point of a given projective line can be seen as the "point at infinity".

For example, take the projective line $\mathcal{L}\Pi$ determined by the plane $\Pi : x + y - z = 0$ in $\mathbb{R}^3$.
Its points are $[a : b : c]$, where $a + b - c = 0$, for example $[1 : 1 : 2]$ or $[3 : 5 : 8]$.

Take the point $[1 : 1 : 2]$ of $\mathcal{L}\Pi$ for example.
$[1 : 1 : 2]$ is the line $\{(t, t, 2t) : t \in \mathbb{R}\}$ through O in $\mathbb{R}^3$.

Take a line L in Π parallel to $[1 : 1 : 2]$, not through O.
$L = \{(2, 1, 3) + t(1, 1, 2)\} = \{[2 + t, 1 + t, 3 + 2t]\}$.



Every point Q of L determines the point OQ or $[2 + t : 1 + t : 3 + 2t]$ of $\mathcal{L}\Pi$. This accounts for all points of $\mathcal{L}\Pi$ except $[1 : 1 : 2]$.

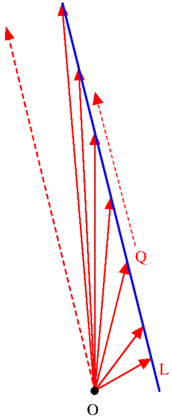
Every point is a "point at infinity"

No point of $\mathbb{RP}^2$ is intrinsically a "point at infinity". Any point of a given projective line can be seen as the "point at infinity".

For example, take the projective line $\mathcal{L}\Pi$ determined by the plane $\Pi : x + y - z = 0$ in $\mathbb{R}^3$.
Its points are $[a : b : c]$, where $a + b - c = 0$, for example $[1 : 1 : 2]$ or $[3 : 5 : 8]$.

Take the point $[1 : 1 : 2]$ of $\mathcal{L}\Pi$ for example.
$[1 : 1 : 2]$ is the line $\{(t, t, 2t) : t \in \mathbb{R}\}$ through O in $\mathbb{R}^3$.

Take a line L in Π parallel to $[1 : 1 : 2]$, not through O.
$L = \{(2, 1, 3) + t(1, 1, 2)\} = \{[2 + t, 1 + t, 3 + 2t]\}$.



As $t \to \pm\infty$, OQ approaches the line through O parallel to L, $[1 : 1 : 2]$.
In this view, $[1 : 1 : 2]$ is the "point at infinity". Every point of $\mathcal{L}\Pi$ appears as the "point at infinity" in $\mathcal{L}\Pi$ for some choice of L in Π.

Incidence of points and lines in $\mathbb{RP}^2$

In $\mathbb{RP}^2$, every pair of distinct points belongs together to a unique line.

This property is shared by the Euclidean plane and the hyperbolic plane, but not the sphere.

Distinct projective points $[a_1 : b_1 : c_1]$ and $[a_2 : b_2 : c_2]$ determine a unique plane Π in $\mathbb{R}_3$ that contains the points $O, P(a_1, b_1, c_1), Q(a_2, b_2, c_2)$. Then $\mathcal{L}\Pi$ is the unique projective line containing $[a_1 : b_1 : c_1]$ and $[a_2 : b_2 : c_2]$ in $\mathbb{RP}^2$.

Every pair of distinct lines in $\mathbb{RP}^2$ intersect in a unique point.

This property is not shared by the Euclidean plane, $\mathcal{H}^2$ or the sphere S^2.
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be distinct projective lines.
Then there are planes Π_1 and Π_2 through O in $\mathbb{R}^3$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively the sets of lines through O in Π_1 and Π_2.
As planes in $\mathbb{R}^3$, Π_1 and Π_2 intersect in a single line through O: this is the point of intersection in $\mathbb{RP}^2$ of the projective lines $\mathcal{L}_1$ and $\mathcal{L}_2$.