Euclidean and non-Euclidean Geometry (MA3101)
Lecture 15: What is Projective Geometry?

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Background from Linear Systems

A system of linear equations is homogeneous if its equations all have constant term 0.

$$\begin{array}{ccccc} x & + & y & = & 0 \\ x & + & 2y & = & 0 \end{array} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A homogeneous system always has a solution ($x_1 = x_2 = \cdots = x_n = 0$), and its set of solutions is a vector subspace of $\mathbb{R}^n$: it is the right nullspace of the coefficient matrix.

An inhomogeneous system is $A_{(m\times n)}x_{(n\times 1)} = b_{(m\times 1)}$ where b is a non-zero vector, A is the coeffecient matrix and x is the vector whose entries are the variable names.

Examples

1. $\begin{array}{ccccc} x & + & y & = & 1 \\ x & + & 2y & = & 1 \end{array} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$ Unique solution $x = 1, y = 0$.
2. $\begin{array}{ccccc} x & + & y & = & 1 \\ x & + & y & = & 2 \end{array} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$ No solution (parallel lines).

Homogeneous and Inhomogeneous Systems

Example $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ has the particular solution $p : x = y = z = 1$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has solution set $W = \left\{ \begin{pmatrix} t \\ -t \\ 0 \end{pmatrix} \right\}$,

a one-dimensional subspace (line) of $\mathbb{R}^3$. The full set of solutions of the first system is

$$p + W = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} t \\ -t \\ 0 \end{pmatrix} \right\}$$

This is an affine subspace of dimension 1 in Euclidean 3-space: a line that does not pass through the origin.

Consistent and inconsistent systems

Suppose the matrix A has a nullspace of dimension k, so that the solution set of the homogeneous system $Ax = 0$ is a vector space W of dimension k. Now suppose $Ax = b$ is an inhomogeneous system (with the same A). There are two possibilities:

1. If the vector p is a particular solution to $Ax = b$, then the solution set of $Ax = b$ is the affine space $p + W$ of dimension k.
2. If $Ax = b$, has no solution, its solution set is empty. We say the system is inconsistent in this case.

An inconsistent system means that the individual equations describe objects that have no common point of intersection in $\mathbb{R}^n$; they are "parallel". But a pair of parallel lines in physical space "appear to intersect at a point at infinity". Can we have a geometric interpretation that incorporates this idea?

Another interpretation

Example A consistent and an inconsistent system:

$$(A) \begin{array}{ccccc} x & + & y & = & 2 \\ x & + & 2y & = & 3 \end{array} \qquad (B) \begin{array}{ccccc} x & + & y & = & 2 \\ x & + & y & = & 3 \end{array}$$

Solution sets are $x = y = 1$ and the empty set.
We introduce a new variable z and consider instead the homogeneous systems

$$(A') \begin{array}{ccccc} x & + & y & = & 2z \\ x & + & 2y & = & 3z \end{array} \qquad (B') \begin{array}{ccccc} x & + & y & = & 2z \\ x & + & y & = & 3z \end{array}$$

Now the first has solution set $\{(t, t, t) : t \in \mathbb{R}\}$.
The second has solution set $\{(t, -t, 0) : t \in \mathbb{R}\}$.

Each of these is a line through the origin in $\mathbb{R}^3$, reflecting the fact that the coefficient matrices have nullity 1.

The Real Projective Plane

If $z' \neq 0$, then (x', y', z') is a solution of (A') if and only if $(\frac{x'}{z'}, \frac{y'}{z'})$ is a solution of (A).

In our example, (x', y', z') is a solution provided $\frac{x'}{z'} = \frac{x'}{y'} = 1$, or $[x' : y' : z'] = [1, 1, 1]$.

Same for (B') and (B), but (B') has no solution with $z \neq 0$.

Solutions with $z = 0$: $x = y = z = 0$ for (A'), and $\{(t, -t, 0)\}$ for (B').

How do we interpret these?

Definition We define an equivalence relation on the set of nonzero vectors in $\mathbb{R}^3$ by

$$(a, b, c) \sim (a', b', c') \iff (a', b', c') = \lambda(a, b, c), \lambda \neq 0, \lambda \in \mathbb{R}.$$

Equivalence classes are lines through the origin in $\mathbb{R}^3$.
These classes are the points of the real projective place $\mathbb{RP}^2$
The class of (a, b, c) is denoted $[a : b : c]$.

Solutions of systems in $\mathbb{RP}^2$

The solution sets of (A') and (B') are respectively the points $[1 : 1 : 1]$ and $[1 : 1 : 0]$ of $\mathbb{RP}^2$.

If $[a : b : c]$ is a solution and $c \neq 0$, then $[a : b : c] = [\frac{a}{c} : \frac{b}{c} : 1]$ in $\mathbb{RP}^2$, and $(\frac{a}{c}, \frac{b}{c})$ is a solution of the original system.

For a solution $[x : y : z]$, we are interested in the ratios $[x : z]$ and $[y : z]$.

To relate the solutions of (B') and (B), we can think of $z = 0$ as representing a limit where $x \to \infty, y \to \infty$.

Then the solution $[1, -1, 0]$ of (B') captures the idea that the two lines of B appear to intersect at infinity.