Euclidean and non-Euclidean Geometry (MA3101)
Lecture 14: Incidence of Lines in the Hyperbolic Plane

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Intersections of lines in $\mathcal{H}^2$

A line in $\mathcal{H}^2$ is the intersection of $\mathbb{H}^2$ ($x^2 + y^2 + z^2 = -1,\ z > 0$) with a plane Π through O in $\mathbb{R}^3$.

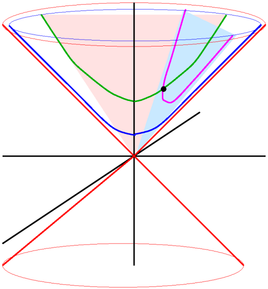
Not all planes in $\mathbb{R}^3$ intersect $\mathcal{H}^2$, only those with a normal vector that points into the region $z^2 < x^2 + y^2$. These determine the lines of $\mathcal{H}^2$.

Let H_1 and H_2 be two lines in $\mathcal{H}^2$. Then

$$H_1 = \Pi_1 \cap \mathcal{H}^2,\ H_2 = \Pi_2 \cap \mathcal{H}^2,$$

where Π_1 and Π_2 are planes through O in $\mathbb{R}^3$.

The planes Π_1 and Π_2 intersect in $\mathbb{R}^3$ in a (Euclidean) line L through O.



There are two possibilities:

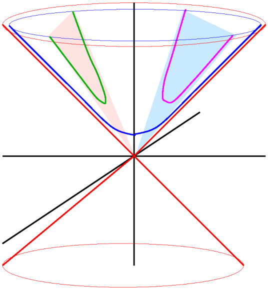
1. L intersects $\mathcal{H}^2$ in exactly one point, and H_1 and H_2 intersect in $\mathcal{H}^2$ at just this point (green and pink lines in the picture).
2. L does not intersect $\mathcal{H}^2$ at all. Then H_1 and H_2 are disjoint.

Non-intersecting lines in $\mathcal{H}^2$.

Let H_1 and H_2 be two lines in $\mathcal{H}^2$. Then

$$H_1 = \Pi_1 \cap \mathcal{H}^2,\ H_2 = \Pi_2 \cap \mathcal{H}^2,$$

where Π_1 and Π_2 are planes through O in $\mathbb{R}^3$.

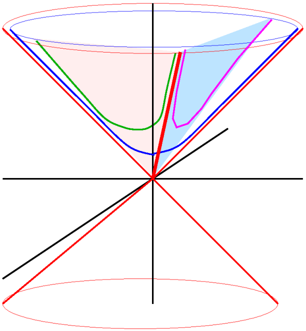


The planes Π_1 and Π_2 intersect in $\mathbb{R}^3$ in a (Euclidean) line L through O.

1. L intersects $\mathcal{H}^2$ if it enters the region $x^2 + y^2 - z^2 < 0$.
2. Otherwise L does not intersect $\mathcal{H}^2$, there are two cases:
   1. If L (except O) is contained in the region $x^2 + y^2 + z^2 > 0$, then H_1 and H_2 don't intersect and diverge (picture above).
   2. If L is contained in the cone $z^2 = x^2 + y^2$ (red in the picture), then H_1 and H_2 approach each other at infinity along the "upper part" of L. In this case, H_1 and H_2 are said to be asymptotically parallel.

The Asymptotically Parallel Case

The bold red line denotes the intersection L of the planes Π_1 and Pi_2, indicated by the pale pink and pale blue triangles. The green and pink lines represent geodesics in $\mathcal{H}^2$ that are asymptotically parallel. They do not intersect in $\mathcal{H}^2$, but at one "end" they both have the line L as an asymptote.



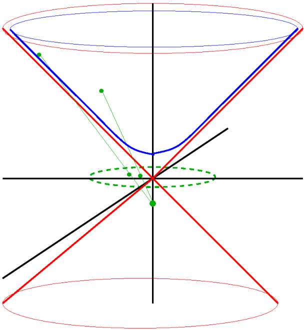
Diverging or being asymptotically parallel are the two possible configurations of a pair of non-intersecting lines in the hyperbolic plane.

How Euclid's Parallel Postulate translates to $\mathcal{H}^2$

Let H be a line in $\mathcal{H}^2$ and let P be a point not belonging to L.
After an isometry of $\mathcal{H}^2$ we can assume that H is $\{(\sinh s, 0, \cosh s)\}$, the intersection of $\mathcal{H}^2$ with the XZ-plane.
The point P is a particular point of $\mathcal{H}^2$ not on H.

1. For every point Q of H, there is a unique line through P in $\mathcal{H}^2$ that intersects H in P. It is the intersection of $\mathcal{H}^2$ with the plane OPQ.
2. There are multiple lines through P in $\mathcal{H}^2$ that diverge from H. Take any vector v in $\mathbb{R}^3$ of the form $(1, 0, \alpha)$, where $|\alpha| < 1$. Then the line L consisting of all scalar multiples of v is contained in the XZ-plane and does not intersect $\mathcal{H}^2$. The plane Π that contains L and P intersects $\mathcal{H}^2$ in a geodesic that diverges from H.
3. Let Π' be the plane containing P and the line L spanned by the vector $(1, 0, 1)$. Then Π' intersects the XZ-plane in the L, which is asymptotic to both H and the geodesic line $H' = \Pi' \cap \mathcal{H}^2$. In this case H and H' are asymptotically parallel.

The hyperbolid $\mathcal{H}^2$ can be mapped to the interior of the unit disc D in the XY-plane as follows.



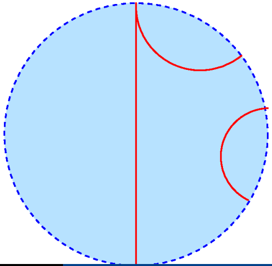
The point P of the hyperboloid is mapped to the point of the XY-plane on the line segment joining P to $(0, 0, -1)$

So $(1, 0, 0) \to O$.

Horizontal cross-sections of the hyperbolid are mapped to circles with centre O.

The geodesics of $\mathcal{H}^2$ are mapped to

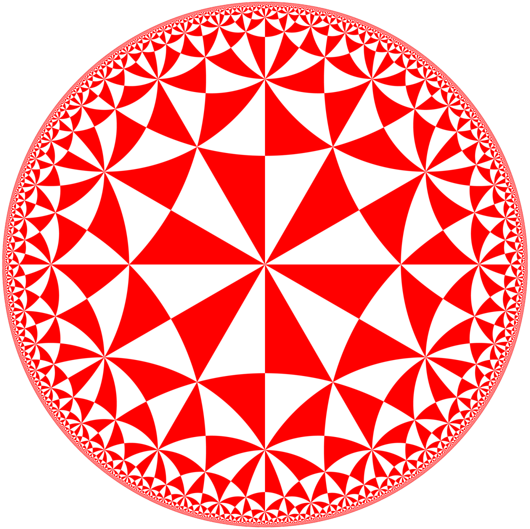
1. Diameters of D
2. Circular arcs in D whose tangents at the unit circle are diameters of D.



Asympotically parallel lines intersect on the boundary (which is not part of the hyperbolic plane).

Tessellations of the Hyperbolic Plane

The triangular tiles in this tessellation are all congruent (same side lengths and area).



Escher's *Circle Limit III, 1959*