Euclidean and non-Euclidean Geometry (MA3101)
Lecture 13: Geodesics in the Hyperbolic Plane

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The hyperbolic plane

Exactly as $S^2 : x^2 + y^2 + z^2 = 1$ is obtained by rotating the unit circle inthe XZ-plane about the Z-axis, we obtain the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$ by rotating $\mathcal{H}^1$ about the Z-axis. The Lorentz inner product for vectors in $\mathbb{R}^3$ is defined by

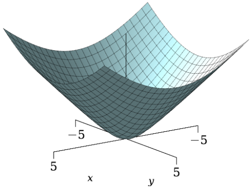
$$\boxed{(x_1, y_1, z_1) \cdot_L (x_2, y_2, z_2) = x_1x_2 + y_1y_2 - z_1z_2}$$

$\mathcal{H}^2$, equipped with the Lorentz pseudometric in which the squared length of a vector v is $v \cdot_L v$, is (the hyperboloid model of) the hyperbolic plane.

$\mathcal{H}$ is the intersection of $\mathcal{H}^2$ with the XZ-plane.

We have a concept of distance in $\mathcal{H}$, defined by

$$d_{\mathcal{H}}(P, Q) = \cosh^{-1}(-P \cdot_L Q).$$



This will extend to $\mathcal{H}^2$.

$\mathcal{H}^2$ is homogeneous and isotropic

1. Rotation R_θ through any θ about the Z-axis preserves $\mathcal{H}^2$ and $\cdot_L$. The matrix of this rotation is $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
   Exercise Check directly that $R_\theta(u) \cdot_L R_\theta(v) = u \cdot_L v$ for all $u, v \in \mathbb{R}^3$.
2. For any $\alpha \in \mathbb{R}$, the translation by α along $\mathcal{H}$ extends to an isometry T_α of $\mathcal{H}^2$, with matrix $\begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha \\ 0 & 1 & 0 \\ \sinh\alpha & 0 & \cosh\alpha \end{pmatrix}$.
   Exercise Check that T_α preserves $\mathcal{H}^2$ and $\cdot_L$. Note T_α sends $(\sinh(-\alpha), 0, \cosh(-\alpha))$ to $(0, 0, 1)$.

Consequence 1 For any point P of $\mathcal{H}^2$, there is an isometry τ of $\mathcal{H}^2$ with $\tau(P) = (0, 0, 1)$. We can apply a rotation about the Z-axis to take P to the XZ-plane, then apply the appropriate T_α.

Consequence 2 All points of $\mathcal{H}^2$ are images of each other under isometries. Like S^2, $\mathcal{H}^2$ is homogeneous. All points “look the same”.

Consequence 3 All directions from $(0, 0, 1)$ are equivalent under the rotation about the Z-axis. So all directions “look the same”.

Geodesics in $\mathcal{H}^2$

Geodesics in S^2 are intersections of S^2 with planes through the origin O.

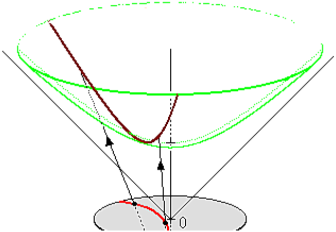
The same is true in the hyperboloid model of $\mathcal{H}^2$.

Theorem Let P and Q be points of $\mathcal{H}^2$. The shortest path (in the hyperbolic distance) from P to Q in $\mathcal{H}^2$ is along the intersection of $\mathcal{H}^2$ with the unique plane in R^2 through P, Q and O.

A plane that contains the points $(0, 0, 1)$ and O intersects $\mathcal{H}^2$ in a rotated copy of $\mathcal{H}$.

Other planes intersect $\mathcal{H}^2$ in curves like the dark red one in this picture.

As for S^2, we apply isometries to move P to $(0, 0, 1)$ and Q to a point in the XZ-plane.



As linear transformations of the ambient space $\mathbb{R}^3$, these moves take the plane OPQ to the plane $Y = 0$, and take the relevant path from P to Q to a segment of $\mathcal{H}$.

Geodesics are intersections with planes

Theorem Let P and Q be points of $\mathcal{H}^2$. The shortest path (in the hyperbolic distance) from P to Q in $\mathcal{H}^2$ is along the intersection of $\mathcal{H}^2$ with the unique plane in R^2 through P, Q and O.

The proof is analogous to the one for S^2 (see Lecture 7).

1. We can assume that $P = (0, 0, 1)$ and $Q = (\sinh\alpha, 0, \cosh\alpha)$.
2. Let $\mathcal{C}$ be any path in $\mathcal{H}^2$ from P to Q, parametrized by t.

   $$\mathcal{C} : t \to (f_1(t), f_2(t), f_3(t)), a \leq t \leq b.$$

   The hyperbolic length of $\mathcal{C}$ is

   $$\int_{t=a}^{t=b} \sqrt{(f_1'(t))^2 + (f_2'(t))^2 - (f_3'(t))^2\,dt}$$

3. To relate f_1, f_2, f_3 to the surface $\mathcal{H}^2$, we note that $\mathcal{H}^2$ is the set of points in $\mathbb{R}^3$ of the form

   $$(\sinh s\cos\theta, \sinh s\sin\theta, \cosh s).$$

   (Reached by rotating $(\sinh s, 0, \cosh s)$ through θ about the Z-axis.)

Geodesics are intersections with planes

Theorem Let P and Q be points of $\mathcal{H}^2$. The shortest path (in the hyperbolic distance) from P to Q in $\mathcal{H}^2$ is along the intersection of $\mathcal{H}^2$ with the unique plane in R^2 through P, Q and O.

The proof is analogous to the one for S^2 (see Lecture 7).

4. Rewriting in terms of s and θ

$$\begin{aligned} f_1(t) = \sinh s(t) \cos\theta(t) \quad &\implies \quad f_1'(t) = \cosh s(t) \cos\theta(t)\dot{s} - \sinh s(t) \sin\theta(t)\dot{\theta} \\ f_2(t) = \sinh s(t) \sin\theta(t) \quad &\implies \quad f_2'(t) = \cosh s(t) \sin\theta(t)\dot{s} + \sinh s(t) \cos\theta(t)\dot{\theta} \\ f_3(t) = \cosh s(t) \quad &\implies \quad f_3'(t) = \sinh s(t)\dot{s}. \end{aligned}$$

5. $(f_1'(t))^2 + (f_2'(t))^2 - (f_3'(t))^2 =$
   $\cosh^2 s(t)\dot{s}^2 + \sinh^2 s(t)\dot{\theta}^2 - \sinh^2 s(t)\dot{s}^2 = \dot{s}^2 + \sinh^2 s(t)\dot{\theta}^2$
6. Now the hyperbolic length of $\mathcal{C}$ is $\int_{t=a}^{t=b} \sqrt{\dot{s}^2 + \sinh^2 s\dot{\theta}^2}\, dt$.
   This is minimized if $\dot{\theta} = 0$ along $\mathcal{C}$, that is if $\mathcal{C}$ is a segment of $\mathcal{H}$.

Intersections of lines in $\mathcal{H}^2$

A line in $\mathcal{H}^2$ is the intersection of $\mathbb{H}^2$ with a plane Π through O in $\mathbb{R}^3$.

Not all planes in $\mathbb{R}^3$ intersect $\mathcal{H}^2$, only those with a normal vector that points into the region $z^2 < x^2 + y^2$. These determine the lines of $\mathcal{H}^2$.

Let H_1 and H_2 be two lines in $\mathcal{H}^2$. Then

$$H_1 = \Pi_1 \cap \mathcal{H}^2,\ H_2 = \Pi_2 \cap \mathcal{H}^2,$$

where Π_1 and Π_2 are planes through O in $\mathbb{R}^3$.

The planes Π_1 and Π_2 intersect in $\mathbb{R}^3$ in a (Euclidean) line L through O.

There are two possibilities:

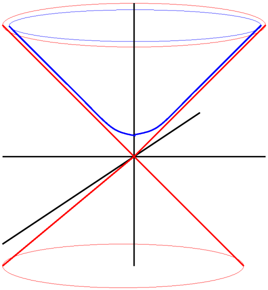
- L intersects $\mathcal{H}^2$ in exactly one point. Then the lines H_1 and H_2 intersect in $\mathcal{H}^2$ in one point. This occurs if a vector in the direction of L points into the region $z^2 - (x^2 + y^2) > 0$.
- L does not intersect $\mathcal{H}^2$. This means that a vector at O in the direction of L points into the region $z^2 - (x^2 + y^2) \leq 0$.

Non-intersecting lines in $\mathcal{H}^2$.

Let H_1 and H_2 be two lines in $\mathcal{H}^2$. Then

$$H_1 = \Pi_1 \cap \mathcal{H}^2, \; H_2 = \Pi_2 \cap \mathcal{H}^2,$$

where Π_1 and Π_2 are planes through O in $\mathbb{R}^3$.



The planes Π_1 and Π_2 intersect in $\mathbb{R}^3$ in a (Euclidean) line L through O.

If L does not intersect $\mathcal{H}^2$, there are two cases:

1. If L is contained in the cone $z^2 = x^2 + y^2$ (red in the picture), then H_1 and H_2 approach each other at infinity along the "upper part" of L. In this case, H_1 and H_2 are said to be ultraparallel.
2. Otherwise, if L does not intersect $\mathcal{H}^2$, H_1 and H_2 are said to diverge