Rings (MA 416) 2024-2025

Assignment 3

Due date: Wednesday November 20

Problems marked with $*$ are for submission. The rest are for independent study and discussion in the tutorials. Please submit solutions via the Canvas page, or in person at the lecture if you prefer paper.

1. Prove that every finite integral domain is a field.
2. A ring R is called *simple* if its only two-sided ideals are the full ring and the zero ideal. Prove that the matrix ring $M_2(\mathbb{F})$ is simple if $\mathbb{F}$ is a field.
3. * For a field $\mathbb{F}$, prove that the ring of upper-triangular matrices in $M_2(\mathbb{F})$ is *not* simple.
4. Show that a commutative ring is simple if and only if it is a field.
5. * Show the the matrix ring $M_2(\mathbb{Z})$ is not simple.
6. Suppose that a and b are distinct elements of an integral domain R, with the property that the principal ideals $\langle a \rangle$ and $\langle b \rangle$ are equal. Show that $a = ub$ for some unit u of R.
7. For a prime p, let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$ of p elements.
   (a) Write down all the quadratic and cubic polynomials in $\mathbb{F}_2[x]$ and determine which are irreducible.
   (b) Write down all the monic irreducible quadratic and cubic polynomials in $\mathbb{F}_3[x]$.
   (c) * Give an example of an irreducible cubic polynomial in $\mathbb{F}_5[x]$. Deduce that there exists a field with 125 $(= 5^3)$ elements.
8. Prove that every prime ideal of a finite commutative ring R is a maximal ideal.
9. Prove that if p is an irreducible element in a unique factorization domain R, then p is prime.
10. Prove that every maximal ideal of a commutative ring is prime, using only the definitions of prime and maximal ideal, and not any information about properties of factor rings.
11. * Give an example of an integral domain R and an irreducible element p of R for which the principal ideal $\langle p \rangle$ of R is not maximal.
12. * The *descending chain condition* (DCC) holds in a ring R if whenever

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

   is a sequence of (two-sided) ideals in R in which each term contains the subsequent one, there is some m for which $I_k = I_m$ for all $m \geq k$. The DCC holds in R if R contains no infinite strictly descending chain of two-sided ideals.
   (a) Show that the DCC does not hold in $\mathbb{Z}$.
   (b) If R is an integral domain in whih the DCC holds, show that R is a field.
   (Hint: if R contains a non-zero, non-unit element, use it to construct an infinite descending chain of ideals.)