MA3101 Euclidean and non-Euclidean Geometry 2024-2025

Assignment 2

Due date: Wednesday November 20

Problems marked with $*$ are for submission. The rest are for independent study and discussion in the tutorials. Please submit solutions via the Canvas page, or in person at the lecture if you prefer paper.

1. Let $\square$ be a *spherical quadrilateral* in S^2, with distinct vertices P, Q, R, S. This means that the edges of $\square$ are arcs of the great circles PQ, QR, RS and SP. Let the internal angles at P, Q, R, S respectively be $\alpha, \beta, \gamma, \theta$.

   (a) Assume that the edges of $\square$ intersect only at the vertices. Show that the area of $\square$ is $\alpha + \beta + \gamma + \theta - 2\pi$.

   (b) * Suppose that the edges QR and SP of the quadrilateral intersect at a point T of their interiors (so the quadrilateral looks like a bow tie). Give an example to show that in this case the area of the quadrilateral is not determined by α, β, γ and θ.

2. * Suppose that a spherical triangle in S^2 has two angles α and β, and an *external* angle δ at the third vertex. Show that the area of the triangle is $\alpha + \beta - \delta$.

3. Let P, Q and R be distinct points of S^2. Show that the spherical distances from P to Q and from P to R are equal if and only if the Euclidean distances from P to Q and from P to R are equal.

4. Let P be a point of S^2 and let ρ be a positive real number with $\rho < \pi$. The spherical circle $\mathcal{C}$ with centre P and radius ρ is the set of points of S^2 whose distance to P along a geodesic in S^2 is ρ.

   Show that $\mathcal{C}$ is a Euclidean circle in $\mathbb{R}^3$, of circumference $2\pi \sin \rho$. Compare this to the statement of how the circumference of a Euclidean circle depends on its radius.

5. For vectors $u = [u_1, u_2, u_3]$ and $v = [v_1, v_2, v_3]$ in $\mathbb{R}^3$, the *cross product* $u \times v$ is defined by

$$u \times v = [u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1].$$

   (a) Show that $u \times v$ is orthogonal to both u and v.

   (b) For non-zero vectors u and v, show that $u \times v$ is the zero vector if and only if u and v are scalar multiples of each other.

   (c) Show that $||u \times v|| = ||u||\,||v|| \sin \theta$, where θ is the angle between u and v.
   (Hint: Show that $||u \times v||^2 = ||u||^2\,||v||^2 - (u \cdot v)^2$.)

6. For three non-zero vectors u, v, w in $\mathbb{R}^3$, show that the absolute value of $w \cdot (u \times v)$ is the volume of the parallelepiped that has u, v, w as three edges (where these vectors are considered as line segments directed from the origin). Show also that $w \cdot (u \times v)$ is the determinant of the matrix that has the entries of w, u, v respectively as its three rows.

7. * Prove the *spherical sine rule*: If PQR is a spherical triangle in S^2, with arc of lengths a, b, c, all at most π, respectively opposite angles $\alpha\beta, \gamma$, then

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}.$$

(**Hint**: As in the proof of the spherical cosine rule, set up a coordinate system with one vertex at the north pole and another in the XZ-plane. Write the coordinates of the three vertices and consider the determinant of the matrix that has these coordinates in its rows. Apply Question 6.)

8. (a) Find the hyperbolic distance between the points $(-\frac{3}{4}, 0, \frac{5}{4})$ to $(0, \frac{15}{8}, \frac{17}{8})$ in $\mathcal{H}^2$.

   (b) * Find the hyperbolic distance between the points $(\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}, \frac{5}{4})$ to $(0, \frac{15}{8}, \frac{17}{8})$ in $\mathcal{H}^2$.

9. (a) Prove that the $P \cdot_L Q$ is negative for any pair of points P and Q of $\mathcal{H}$, where $\mathcal{H}$ is the set of points of $\mathbb{R}^2$ whose coordinates satisfy $X^2 - Y^2 = -1$.

   (b) * Prove that $P \cdot_L Q$ is negative for any pair of points P and Q of the hyperbolic plane $\mathcal{H}^2$, where $\mathcal{H}^2$ is the set of points of $\mathbb{R}^3$ whose coordinates satisfy $X^2 + Y^2 - Z^2 = -1$.

10. (a) For $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$. Show that T_α preserves the Lorentz inner product on $\mathbb{R}^2$, i.e. $T_\alpha(u) \cdot_L T_\alpha(v) = u \cdot_L v$ for all $u, v \in \mathbb{R}^2$.

    (b) For $\alpha \in \mathbb{R}$, let $S_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $\begin{pmatrix} -\cosh\alpha & \sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}$. Show that S_α preserves the Lorentz inner product on $\mathbb{R}^2$, i.e. $S_\alpha(u) \cdot_L S_\alpha(v) = u \cdot_L v$ for all $u, v \in \mathbb{R}^2$.

    (c) Deduce that T_α and S_α preserve $\mathcal{H}$, i.e. that $T_\alpha(u)$ and $S_\alpha(u)$ belong to $\mathcal{H}$ whenever $u \in \mathcal{H}$.

11. *

    (a) For $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with standard matrix $\begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha \\ 0 & 1 & 0 \\ \sinh\alpha & 0 & \cosh\alpha \end{pmatrix}$. Show that T_α preserves the Lorentz inner product on $\mathbb{R}^2$, i.e. $T_\alpha(u) \cdot_L T_\alpha(v) = u \cdot_L v$ for all $u, v \in \mathbb{R}^3$.

    (b) For $\theta \in \mathbb{R}$, let $S_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with standard matrix $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that S_θ preserves the Lorentz inner product on $\mathbb{R}^2$, i.e. $S_\theta(u) \cdot_L S_\alpha(v) = u \cdot_L v$ for all $u, v \in \mathbb{R}^3$.

    (c) Deduce that T_α and S_θ preserve $\mathcal{H}^2$, i.e. that $T_\alpha(u)$ and $S_\alpha(u)$ belong to $\mathcal{H}$ whenever $u \in \mathcal{H}^2$.