Euclidean and non-Euclidean Geometry (MA3101)
Lecture 9: Curvature

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Curvature

The statement that $\alpha + \beta + \gamma - \pi$ is the area of a triangle is a special case of the Gauss-Bonnet Theorem, which expresses the difference between π and the interior angle sum in a geodesic triangle in terms of the curvature of the surface.

The Euclidean plane has curvature 0, there the angle sum is just π.

The sphere S^2 has constant curvature 1 (more on that soon), there the excess $\alpha + \beta + \gamma - \pi$ is the area of the triangle.

Thinking about a surface where the sum of the internal angles in a geodesic triangle is less than π will lead to hyperbolic geometry.

Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^3$ (or $\mathbb{R}^2$) with a parametric description

$$\{f_1(t), f_2(t), f_3(t)\}$$

The vector $T = (\dot{f_1}(t), \dot{f_2}(t), \dot{f_3}(t))$ is a tangent vector to T. Its direction (at any value of t) is the instantaneous direction of motion of a traveller along the curve in the direction of increasing t.

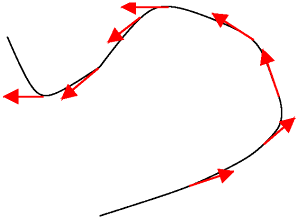
The unit tangent vector is $\mathbf{T} = \frac{T}{||T||}$, a continuous function of t.

Example $\mathcal{C} : t \to (t, t^2, t^3)$, $t \geq 0$, a curve in $\mathbb{R}^3$

$T = (1, 2t, 3t^2)$, $\mathbf{T} = \frac{1}{\sqrt{1+4t^2+9t^4}}(1, 2t, 3t^2)$

At $(1, 1, 1)$ $(t = 1)$, $\mathbf{T} = \frac{1}{\sqrt{14}}(1, 2, 3)$,

At $(1, 4, 8)$ $(t = 2)$, $\mathbf{T} = \frac{1}{\sqrt{161}}(2, 4, 16)$.

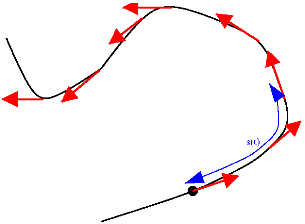


Curvature of a plane curve

A straight line in the plane is not curvy.
A circle of radius 1 is more curvy than a circle of radius 10, but not as curvy as one of radius 1.
How do we quantify this sense of "curviness"?

Let s (a function of t) denote distance travelled along the curve, as t increases from an agreed starting point. The unit tangent vector $\mathbf{T}$ changes in direction not in length with s. The curvature κ (or $\kappa(t)$) of $\mathcal{C}$ is defined as

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$



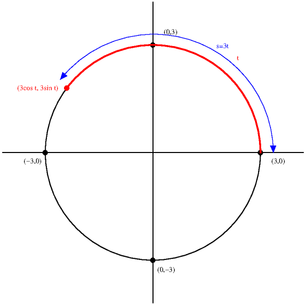
A line in the plane has constant curvature 0, its unit tangent vector never changes direction.

Curvature of a circle

Example: a circle of radius 3

$\mathcal{C} : t \to (\underbrace{3\cos t}_{f_1(t)}, \underbrace{3\sin t}_{f_2(t)}),\ t \geq 0$

Tangent vector
$T = (\dot{f_1}, \dot{f_2}) = (-3\sin t, 3\cos t)$
Unit tangent:
$\mathbf{T} = \frac{1}{3}(-3\sin t, 3\cos t) = (-\sin t, \cos t)$.
Distance along curve: $s(t) = 3t$, $t = \frac{1}{3}s$
(the circle is a special easy example here!)



Curvature

$$\kappa(t) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}}{dt}\frac{dt}{ds} \right\| = \left\| (-\cos t, -\sin t) \times \frac{1}{3} \right\| = \frac{1}{3}.$$

The curvature of a circle of radius R is $\frac{1}{R}$

The unit circle has (constant) curvature 1.

The normal vector and tangent plane to a surface

We'd like to extend the concept of curvature from curves to (smooth) surfaces and then assert that the plane has (constant) curvature 0, and that the unit sphere has (constant) curvature $+1$.

Let S be a smooth surface in $\mathbb{R}^3$ (no corners).

Let Q be a point in S.

There are many smooth curves in S through Q.
Somehow the curvature of the surface at Q will be defined in terms of their curvatures.

Every curve in S through P has a tangent vector at Q.

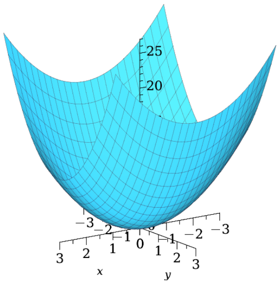
Claim All of these tangent vectors at Q to curves in S lie in a plane, called the tangent plane to S at Q.

Being a plane in $\mathbb{R}^3$, the tangent plane to S at Q has a normal direction.

Example: $S : z = x^2 + 2y^2$ (a paraboloid)

Find the equation of the tangent plane to the paraboloid S at the point $Q : (1, 2, 9)$.

Write $f(x, y, z) = x^2 + 2y^2 - z$.
$S : f(x, y, z) = 0$.
Gradient of f:
$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 4y, -1)$.
$\nabla(f)|_{(1,2,9)} = (2, 8, -1)$. This is the normal vector to P.



Equation of P: $\boxed{(2, 8, -1) \cdot (x - 1, y - 2, z - 9) = 0, \; 2x + 8y - z = 9.}$

If a smooth surface S has equation $f(x, y, z) =$constant, the equation of the tangent plane P to S at (x_0, y_0, z_0) is

$$P : \nabla f \cdot (x - x_0, y - y_0, z - z_0) = 0$$

What happened there?

Let S be a smooth surface in $\mathbb{R}^3$, with equation $f(x, y, z) =$constant.

Let $\mathcal{C}$ be a curve in S, parametrized by $t \to (f_1(t), f_2(t), f_3(t))$.

The tangent vector to $\mathcal{C}$ is $T = (f_1'(t), f_2'(t), f_3'(t))$.

On $\mathcal{C}$, f is a function of t via the dependence of x, y, z on t, and it is constant. Hence (on $\mathcal{C}$)

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot T.$$

This is saying that at any point $Q = (x_0, y_0, z_0)$ where S is smooth, ∇f (at Q) is orthogonal to the tangent vector at Q of every curve in S that passes through Q. So ∇f is a *normal vector* to the tangent plane $T_Q(S)$ of S at Q.

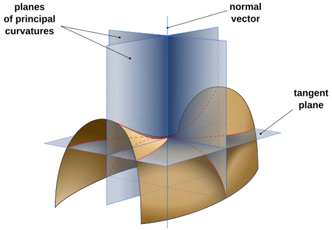
Definition ∇f (at Q) is called a normal vector to the surface S at the point Q. A unit normal vector is a unit vector in the direction of ∇f.

Normal Sections

On a smooth surface S, the (unit) normal vector at a point may have either of two opposite directions. We choose one of these (at a particular point) to be the unit normal vector, and extend the designation by requiring that the unit normal vector should vary continuously as we travel around the surface (so it doesn't abruptly reverse direction).

This amounts to choosing one "side" of the surface for the normal vector to point into, and it works provided that S is orientable - it fails on a Möbius band which is non-orientable.

At a point Q of S, let $\mathbf{n}$ be the unit normal.
Let L be the line through Q in the direction of $\mathbf{n}$.
Every plane in $\mathbb{R}^3$ that contains L intersects S in a curve.



picture courtesy of Wikipedia

These curves are the normal sections of S at Q.