Euclidean and non-Euclidean Geometry (MA3101)
Lecture 8: Area of a Spherical Triangle

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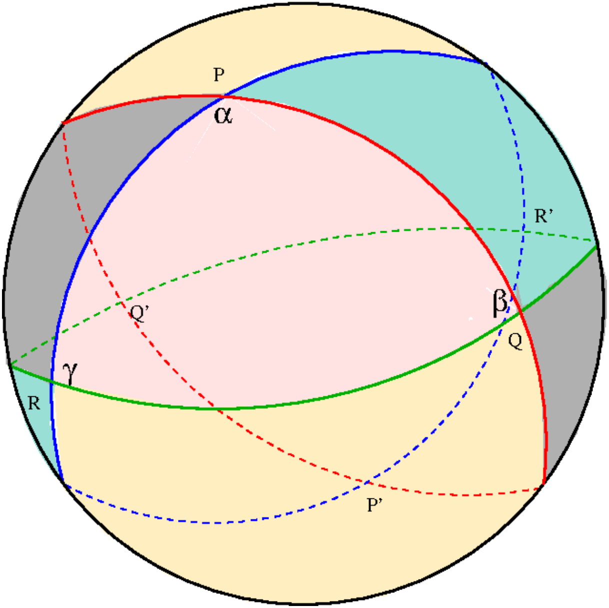
$$\boxed{\text{Area}\triangle = \alpha + \beta + \gamma - \pi}$$

Theorem Let $\triangle$ be a triangle in S^2, with geodesic edges, and internal angles α, β, γ at vertices P, Q, R respectively. Then the area of the triangle T is $\alpha + \beta + \gamma - \pi$.

Notes

1. This is saying that the sum of the internal angles in a spherical triangle is always at least π, and equal to π only when the triangle is degenerate (all vertices collinear).
2. It is also saying that the sum of the internal angles in a spherical triangle is not the same for every triangle, unlike in the plane.

The picture of the proof



The proof

$$\boxed{\text{Area}(\triangle PQR) = \alpha + \beta + \gamma - \pi}$$

Write A for the surface area of S^2 ($A = 4\pi$), and T for the area of the triangle $\triangle$.

Colour the edges of the triangle green, blue and red, respectively opposite the angles α, β and γ. Extend these edges (with colours) to great circles on the sphere.

The pairs of great circles intersect again at the antipodes p', Q', R' of the vertices P, Q, R, enclosing a triangle $\triangle'$ with the same area as $\triangle$.

The blue and red circle divide the surface into four regions, all intersecting at the vertex P and its antipode P'.

One of these four includes $\triangle$. Colour this region, and the opposite one (also with angle α) YELLOW.

The proof

$$\boxed{\text{Area}(\triangle PQR) = \alpha + \beta + \gamma - \pi}$$

- Colour the region that includes $\triangle$ between the blue and red great circles, and the opposite one (both with angle α) YELLOW. The yellow area is $\frac{2\alpha}{2\pi}A$.
- Colour the region that includes $\triangle$ between the green and red great circles, and the opposite one (both with angle β) GREY. The grey area is $\frac{2\beta}{2\pi}A$.
- Colour the region that includes $\triangle$ between the green and blue great circles, and the opposite one (both with angle γ) LIGHT BLUE. The light blue area is $\frac{2\gamma}{2\pi}A$.

Now the interior of $\triangle PQR$, and the interior of $\triangle P'Q'R'$ have been coloured with all three colours. Every other point on the surface of S^2 has been coloured with exactly one of them.

The proof

$$\boxed{\text{Area}(\triangle PQR) = \alpha + \beta + \gamma - \pi}$$

Now the interior of $\triangle PQR$, and the interior of $\triangle P'Q'R'$ have been coloured with all three colours. Every other point on the surface of S^2 has been coloured with exactly one of them.

$$\begin{aligned}
\begin{matrix}\text{YELLOW}\\ \text{AREA}\end{matrix} + \begin{matrix}\text{GREY}\\ \text{AREA}\end{matrix} + \begin{matrix}\text{BLUE}\\ \text{AREA}\end{matrix} &= A + 2(\triangle PQR) + 2(\triangle P'Q'R') \\
\Longrightarrow \frac{2\alpha}{2\pi}A + \frac{2\beta}{2\pi}A + \frac{2\gamma}{2\pi}A &= A + 4T \\
\Longrightarrow \frac{A}{\pi}(\alpha + \beta + \gamma) &= A + 4T
\end{aligned}$$

Since the surface area A of S^2 is 4π, we conclude

$$T = \alpha + \beta + \gamma - \pi.$$

Curvature

The statement that $\alpha + \beta + \gamma - \pi$ is the area of a triangle is a special case of the Gauss-Bonnet Theorem, which expresses the difference between π and the interior angle sum in a geodesic triangle in terms of the curvature of the surface.

The Euclidean plane has curvature 0, there the angle sum is just π.

The sphere S^2 has constant curvature 1 (more on that soon), there the excess $\alpha + \beta + \gamma - \pi$ is the area of the triangle.

Thinking about a surface where the sum of the internal angles in a geodesic triangle is less than π will lead to hyperbolic geometry.

The unit tangent vector

Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^3$ (or $\mathbb{R}^2$) with a parametric description

$$\{f_1(t), f_2(t), f_3(t)\}$$

The vector $T = (\dot{f}_1(t), \dot{f}_2(t), \dot{f}_3(t))$ is a tangent vector to T. Its direction (at any value of t) is the instantaneous direction of motion of a traveller along the curve in the direction of increasing t.

The unit tangent vector is $\mathbf{T} = \frac{T}{||T||}$, a continuous function of t.

Example $\mathcal{C} : t- > (t, t^2, t^3), t \geq 0$, a curve in $\mathbb{R}^3$

$T = (1, 2t, 3t^2)$, $\mathbf{T} = \frac{1}{\sqrt{1+4t^2+9t^4}}(1, 2t, 3t^2)$

At $(1, 1, 1)$ $(t = 1)$, $\mathbf{T} = \frac{1}{\sqrt{14}}(1, 2, 3)$,

At $(1, 4, 8)$ $(t = 2)$, $\mathbf{T} = \frac{1}{\sqrt{161}}(2, 4, 16)$.