Euclidean and non-Euclidean Geometry (MA3101)
Lecture 7: Arc Length

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Consequences of the spherical cosine rule

$$\boxed{\cos a = \cos\alpha \sin b \sin c + \cos b \cos c}$$

1. The Spherical Triangle Inequality In a spherical triangle PQR whose arc lengths a, b, c are all at most π, $a \leq b + c$, and $a = b + c$ only if all three vertices are on the same great circle.
   Proof Use the trigonometric identity

   $$\cos(b + c) = \cos b \cos c - \sin b \sin c.$$

   Since $\sin b$ and $\sin c$ are positive for $b, c \in (0, \pi)$, this means

   $$\cos a \geq \cos(b + c),$$

   and $\cos a = \cos(b + c)$ only if $\alpha = \pi$ which is the case of "spherical collinearity". Since $a \in [0, \pi]$ and the cosine function is strictly decreasing on this interval, it follows that $a \leq b + c$.
2. The Spherical Pythagorean Theorem If $\alpha = \frac{\pi}{2}$ then $\cos\alpha = 0$ and

   $$\cos a = \cos b \cos c.$$

Geodesic arcs are shortest paths

We don't need to deduce the spherical triangle inequality as a consequence of the cosine rule, if we believe the following theorem from Lecture 3.

Theorem The shortest path on the sphere from P to Q is along the great circle that contains P and Q, for distinct, non-antipodal points P and Q. For antipodal P and Q, we just replace "the great circle" with "a great circle".

To prove this theorem, we need a way of describing lengths of curves on S^2, that allows us to compare them as quantities.

Parametrically defined curves in $\mathbb{R}^3$

Suppose that a curve in $\mathbb{R}^3$ is described in parametric form as the set of points

$$(f_1(t), f_2(t), f_3(t)),\ a \leq t \leq b.$$

Examples

1. The XZ-plane intersects S^2 in the great circle through $(1,0,0)$ and $(0,0,1)$, given by

$$\{(\cos t, 0, \sin t) : 0 \leq t < 2\pi\}.$$

2. The curve

$$\{(t, t, 2t^2) : t \geq 0\}$$

consists of all points in the plane $X = Y$ that satisfy $Z = 2X^2$ $(= 2Y^2)$. It is (part of) a parabola. The set of *all* points of the form $(t, t, 2t^2)$, with $t \in \mathbb{R}$, is the intersection of the paraboloid $Z = X^2 + Y^2$ with the plane $X = Y$.

Length of a parametric curve

Suppose a curve $\mathcal{C}$ is described by the points $(f_1(t), f_2(t), f_3(t))$, $a \leq t \leq b$, and that $\mathcal{C}$ is traversed *once* as t increases from a to b.
The structure *length* of $\mathcal{C}$ is estimated by:

- Divide the interval $[a, b]$ into N subintervals of length Δt.
- Write $t_0, \dots, t_N$ for the successive values t at the endpoints of these subintervals.
- Estimate the distance of the corresponding "subcurves" as if they are line segments. The segment corresponding to the interval $[t_i, t_{i+1}]$ has length

$$||(f_1(t_i + \Delta t) - f_1(t_i), f_2(t_i + \Delta t) - f_2(t_i), f_3(t_i + \Delta t) - f_3(t_i))||$$

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$$\sqrt{\sum_{j=1,2,3} \left(f_j(t_i + \Delta t) - f_j(t_i)\right)^2}$$

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- Estimate the distance of the corresponding "subcurves" as if they are line segments. The segment corresponding to the interval $[t_i, t_{i+1}]$ has length The length of the curve is estimated by

$$\sum_{i=0}^{N-1} \sqrt{\sum_{j=1,2,3} \frac{(f_j(t_i + \Delta t) - f_j(t_i))^2}{(\Delta t)^2}} \Delta t$$

Length of a parametric curve

Suppose a curve $\mathcal{C}$ is described by the points $(f_1(t), f_2(t), f_3(t))$, $a \leq t \leq b$, and that $\mathcal{C}$ is traversed *once* as t increases from a to b. The structure *length* of $\mathcal{C}$ is estimated by:

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- Estimate the distance of the corresponding "subcurves" as if they are line segments. The segment corresponding to the interval $[t_i, t_{i+1}]$ has length In the limit as $\Delta t \to 0$, $N \to \infty$ (provided that the f_j are differentiable functions of t, this gives the following formula for the length $L(\mathcal{C})$ of $\mathcal{C}$

$$L(\mathcal{C}) = \int_{t=a}^{t=b} \sqrt{(f_1'(t))^2 + (f_2'(t))^2 + (f_3'(t))^2}\, dt.$$

Example

Let $\mathcal{C}$ be defined as $\{(t, t^2, \frac{2}{3}t^3), 0 \leq t \leq 1\}$. Find the length of $\mathcal{C}$.

Solution

$f_1(t) = t,\ f_1'(t) = 1, \quad f_2(t) = t^2,\ f_2'(t) = 2t, \quad f_3(t) = \frac{2}{3}t^3,\ f_3'(t) = 2t^2.$

$$\begin{aligned} L(\mathcal{C}) &= \int_{t=0}^{t=1} \sqrt{(f_1'(t))^2 + (f_2'(t))^2 + (f_3'(t))^2}\, dt \\ &= \int_{t=0}^{t=1} \sqrt{1 + 4t^2 + 4t^4}\, dt \\ &= \int_{t=0}^{t=1} \sqrt{(1+2t^2)^2}\, dt = \int_{t=0}^{t=1} 1 + 2t^2\, dt \\ &= t + \frac{2}{3}t^3 \Big|_{t=0}^{t=1} = \frac{5}{3} \end{aligned}$$

Remark This example is carefully chosen, generally we do not expect $\sqrt{(f_1'(t))^2 + (f_2'(t))^2 + (f_3'(t))^2}$ to have a nice antiderivative, even for relatively nice functions f_j. Another case where everything is nice (even nicer) is a circular arc parametrized as $\{(\cos t, \sin t), a \leq t \leq b\}$.

Shortest paths on S^2 - back to the theorem

To prove: For (distinct, non-antipodal) points P and Q on S^2, the shortest path on S^2 from P to A is along the shorter great circle arc that connects them.

We set up the coordinate axes so that $P = (0, 0, 1)$ (the North Pole).
Goal: show the shortest path from P to Q is the meridian through Q.

Think of the meridian through $(1, 0, 0)$ as "zero longitude" and measure longitude from there, as an arc/angle in the unit circle in the XY-plane.

Points on S^2 are described by their latitude θ and longitude ϕ.

Any path from P to Q can be parametrized by functions $\theta(t)$ and $\phi(t)$, describing the latitude and longitude of points along the path, as t increases from 0 to some b.

We want to show that the shortest path is the one of constant longitude, where $\phi(t)$ is a constant function of t.

To apply the arc length formula, we need Cartesian coordinates.

Longitude and Latitude to Cartesian coordinates

Lemma The point of S^2 with latitude θ and longitude ϕ has Cartesian coordinates

$$(\cos\theta\ \cos\phi, \cos\theta\ \sin\phi, \sin\theta)$$

(Check that these satisfy $X^2 + Y^2 + Z^2 = 1$.)

Proof The points of S^2 at latitude θ form a circle $\mathcal{C}$ of radius $\cos\theta$ in the horizontal plane $Z = \sin\theta$.
The point of S^2 in the XY-plane ($Z = 0$) with longitude ϕ is $(\cos\phi, \sin\phi, 0)$.
To get the X- and Y-coordinates of the point with the same longitude at latitude θ, scale the X- and Y- coordinates by the radius $\cos\theta$ of $\mathcal{C}$.
So latitude θ, longitude ϕ translates to the Cartesian coordinates

$$(\cos\theta\ \cos\phi, \cos\theta\ \sin\phi, \sin\theta).$$

The Shortest Path

Now a path from P to Q has a parametric description

$$(\cos\theta(t)\cos\phi(t), \cos\theta(t)\sin\phi(t), \sin\theta(t)), \ \ a \leq t \leq b.$$

The t-derivatives of the components are

- $\dot{x}(t) = -sin\theta(t)\cos\phi(t)\dot{\theta}(t) - \cos\theta(t)\sin\phi(t)\dot{\phi}(t)$
- $\dot{y}(t) = -sin\theta(t)\sin\phi(t)\dot{\theta}(t) + \cos\theta(t)\cos\phi(t)\dot{\phi}(t)$
- $\dot{z}(t) = \cos\theta(t)\dot{\theta}(t)$

Now (check that)

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{\theta}(t)^2 + \cos^2\theta(t)\dot{\phi}(t)^2.$$

The path has length

$$\int_{t=a}^{t=b} \sqrt{\dot{\theta}(t)^2 + \underbrace{\cos^2\theta(t)\dot{\phi}(t)^2}_{\geq 0}}\, dt.$$

This is minimized when $\dot{\phi}(t) = 0$ (i.e. when $\phi(t)$ is constant), and then its value is $\theta(b) - \theta(a)$, the length of the great circle arc PQ.