Euclidean and non-Euclidean Geometry (MA3101)
Lecture 12: The Hyperbolic Plane $\mathcal{H}^2$

Dr Rachel Quinlan

October 30, 2024

Hyperbolic Geometry

From the geometry of $\mathcal{H}$ we will construct a surface $\mathcal{H}^2$ with the following properties in common with the sphere S^2 and Euclidean plane

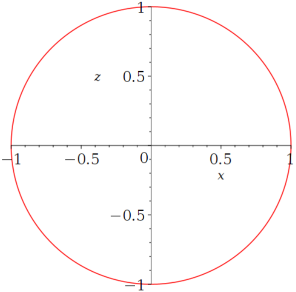
- Every pair of points in $\mathcal{H}^2$ belong to a unique line (and are connected by a unique line segment).
- The geometry is homogenous (the same at every point)
- The geometry is isotropic (the same in all directions)

The three geometries have different incidence behaviour for lines:
Given a line L and a point P not belonging to L

- In the Euclidean plane, there is a unique line through P that does not intersect L
- In S^2, every line through P intersects L (in two points)
- In $\mathcal{H}^2$, there are multiple lines through P that do not intersect L (in different ways)

Isometries of S^1

Definition An isometry of S^1 is a function from S^1 to S^1 that preserves distance along arcs. This means: if a, b are points of S^1 with images a', b', then $\text{distance}_{S^1}(a, b) = \text{distance}_{S^1}(a', b')$. Equivalently $a \cdot b = a' \cdot b'$, if points of S^1 are considered as unit vectors in $\mathbb{R}^2$.



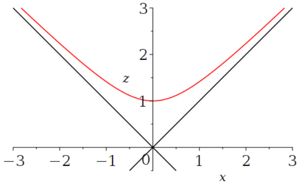
There are two types of isometries of S^1: rotations about the origin through any angle, and reflections in any diameter. Both arise from linear transformations of $\mathbb{R}^2$ with standard matrices as follows:

- Rotation R_α: $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $(\cos t, \sin t) \to (\cos(t+\alpha), \sin(t+\alpha))$
- Reflection M_α in the diameter through $(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2})$:

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \ (\cos t, \sin t) \to (\cos(\alpha - t), \sin(\alpha - t))$$

All these matrices comprise the orthogonal group $O(2)$.

Definition An **isometry** of $\mathcal{H}$ is a function from $\mathcal{H}$ to $\mathcal{H}$ that preserves hyperbolic (Lorentz) distance along arcs. This means: if a, b are points of $\mathcal{H}$ with images a', b', then $\text{distance}_{\mathcal{H}}(a, b) = \text{distance}_{\mathcal{H}}(a', b')$. Equivalently $a \cdot_L b = a' \cdot_L b'$.



One isometry of $\mathcal{H}$ is the **Lorentz translation** T_α, which maps $(\sinh s, \cosh s)$ to $(\sinh(s+\alpha), \cosh(s+\alpha))$ for a fixed $\alpha \in \mathbb{R}$.

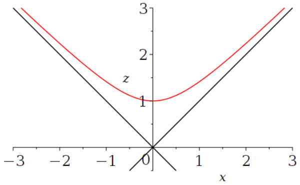
The distance in $\mathcal{H}$ between $(\sinh s_1, \cosh s_1)$ and $(\sinh s_2, \cosh s_2)$ is $|s_1 - s_2|$, the same as $|(s_1+\alpha) - (s_2+\alpha)|$.

The T_α are analogous to rotations in S^1, and also arise from linear transformations of $\mathbb{R}^2$.

The standard matrix of $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is $\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$

Isometries of $\mathcal{H}$

Definition An isometry of $\mathcal{H}$ is a function from $\mathcal{H}$ to $\mathcal{H}$ that preserves hyperbolic (Lorentz) distance along arcs. This means: if a, b are points of $\mathcal{H}$ with images a', b', then $\text{distance}_{\mathcal{H}}(a, b) = \text{distance}_{\mathcal{H}}(a', b')$. Equivalently $a \cdot_L b = a' \cdot_L b'$.



The reflection M_0 in $(0, 1)$, $\mathcal{H} \to \mathcal{H}$, $(\sinh s, \cosh s) \to (-\sinh s, \cosh s)$.

The reflection of $\mathbb{R}^2$ in the Z-axis has matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

For $\alpha \in \mathbb{R}$, the reflection M_α of $\mathcal{H}$ fixes $(\sinh \frac{\alpha}{2}, \cosh \frac{\alpha}{2})$ and maps

$$(\sinh s, \cosh s) \to (\sinh(\alpha - s), \cosh(\alpha - s)) \,.$$

M_α preserves distance in $\mathcal{H}$ and has matrix $\begin{pmatrix} -\cosh \alpha & \sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix}$.

Linear Algebra and/or Group Theory Interpretation

The set of matrices T_α and M_α describing isometries of $\mathcal{H}$ also forms a group, known as the Lorentz group. For a pair of vectors $u = \binom{a}{b}$ and $v = \binom{c}{d}$ in $\mathbb{R}^2$ their ordinary and Lorentz inner products are respectively given by the matrix products

- $u \cdot v \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = u^T I_2 v = u^T v.$
- $u \cdot_L v \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{L_2} \begin{pmatrix} c \\ d \end{pmatrix} = u^T L_2 v.$

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the ordinary scalar product. Then for any vectors $u, v \in \mathbb{R}^2$, $T(u) \cdot T(v) = u \cdot v$.
If M_T is the matrix of T, this means for all u and v that

$$(Mu)^T I_2 (Mv) = u^T I_2 v \Longrightarrow u^T M^T M v = u^T v,$$

for all $u, v \in \mathbb{R}^2$. This means $M^T M = I_2$, so $M^{-1} = M^T$.

The hyperbolic plane

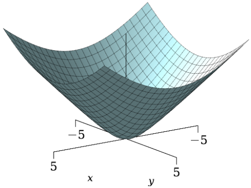
Exactly as $S^2 : x^2 + y^2 + z^2 = 1$ is obtained by rotating the unit circle inthe XZ-plane about the Z-axis, we obtain the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$ by rotating $\mathcal{H}^1$ about the Z-axis.
The Lorentz inner product for vectors in $\mathbb{R}^3$ is defined by

$$\boxed{(x_1, y_1, z_1) \cdot_L (x_2, y_2, z_2) = x_1x_2 + y_1y_2 - z_1z_2}$$

$\mathcal{H}^2$, equipped with the Lorentz pseudometric in which the squared length of a vector v is $v \cdot_L v$, is (the hyperboloid model of) the hyperbolic plane.

$\mathcal{H}$ is the intersection of $\mathcal{H}^2$ with the XZ-plane.
We have a concept of distance in $\mathcal{H}$, defined by

$$d_{\mathcal{H}}(P, Q) = \cosh^{-1}(-P \cdot_L Q).$$



This will extend to $\mathcal{H}^2$.