Euclidean and non-Euclidean Geometry (MA3101)
Lecture 11: Introduction to Hyperbolic Geometry

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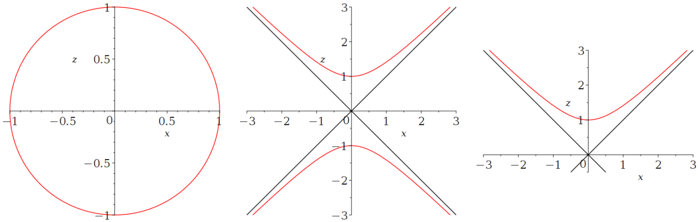
The hyperbola $-x^2 + z^2 = 1$ in $\mathbb{R}^2$

The picture below show the unit circle $x^2 + z^2 = 1$, the hyperbola $-x^2 + z^2 = 1$ (along with its asymptotes $y = \pm x$) and the upper branch $\mathcal{H}$: $z = \sqrt{x^2 - 1}$ of the hyperbola (this is the part that we will care about).

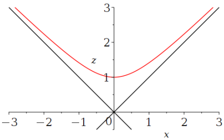


How to parametrize the (upper branch of the) hyperbola?
Its points all have positive z-coordinate. For any positive a,

$$-\left(a - \frac{1}{a}\right)^2 + \left(a + \frac{1}{a}\right)^2 = 4,$$

so $\left(\frac{1}{2}\left(a - \frac{1}{a}\right)\right), \left(\frac{1}{2}\left(a + \frac{1}{a}\right)\right)$ is a point of $\mathcal{H}$.

Parametrization of $\mathcal{H}$



For $a > 0$, the point with coordinates

$$\left(\frac{1}{2}\left(a - \frac{1}{a}\right), \frac{1}{2}\left(a + \frac{1}{a}\right)\right)$$

belongs to $\mathcal{H}$.

$a - \frac{1}{a}$ is a continuous increasing function of a for $a > 0$ (check this).

$$a - \frac{1}{a} \to -\infty \text{ as } a \to 0^+, \text{ and } a - \frac{1}{a} \to \infty \text{ as } a \to \infty.$$

It follows that every point of $\mathcal{H}$ has these coordinates for some a.
Now write $a = e^s$ (we can do this since a is positive). Then $\mathcal{H}$ consists of all points of the form

$$\left(\frac{e^s - e^{-s}}{2}, \frac{e^s + e^{-s}}{2}\right) = (\sinh s, \cosh s), s \in \mathbb{R}.$$

The Hyperbolic Trigonometric Functions

Definition The hyperbolic sine and cosine functions (usually pronounced "sinch" and "cosh") are defined for $s \in \mathbb{R}$ by

$$\sinh s = \frac{1}{2}(e^s - e^{-s}), \ \cosh s = \frac{1}{2}(e^s + e^{-s}).$$

Properties

1. sinh and cosh describe coordinates of points of the unit hyperbola, the way that sin and cos describe points on the circle.
2. $\cosh(-s) = \cosh(s)$ and $\sinh(-s) = -\sinh(s)$, for $s \in \mathbb{R}$.
3. $\sinh^2 s - \cosh^2 s = -1$ (or $\cosh^2 s - \sinh^2 s = 1$), for all $s \in \mathbb{R}$.
4. $\frac{d}{ds}(\sinh s) = \cosh s$ and $\frac{d}{ds}(\cosh s) = \sinh s$.
   This is one rationale for using coordinates cosh and sinh instead of the previously suggested $\frac{1}{2}(a \pm \frac{1}{a})$ for $a > 0$.
5. For $s, t \in \mathbb{R}$, $\cosh s \cosh t - \sinh s \sinh t = \cosh(t - s)$.
6. And many more analogues of trigonometric identities, that we don't need too urgently.

All of these can be proved directly from the definitions.

$\mathcal{H}$ and S^1 (the unit circle) in the xz-plane

The unit circle S^1 in has equation $x^2 + z^2 = 1$ or $z^2 = 1 - x^2$.
Its ambient space is $\mathbb{R}^2$ with the Euclidean metric and ordinary scalar product:

$$(x_1,z_1)\cdot(x_2,z_2)=x_1x_2+z_1z_2,\ ||(x,z)||^2=(x,z)\cdot(x,z),\text{distance}(u,v)=||u-v||.$$

S^1 is parametrized by the trigonometric functions: $x = \cos t$ and $z = \sin t$. The distance along S^1 from $P(\cos\alpha, \sin\alpha)$ to $Q(\cos(\alpha \pm t), \sin(\alpha \pm t))$ is $t = \cos^{-1}(P \cdot Q)$ (provided $t < \pi$).

$\mathcal{H}$ is analogous to S^1, with equation $z^2 = 1 + x^2$ (and $z > 0$).
The expression $x^2 - z^2$ determines the Lorentz pseudometric and the Lorentz scalar product defined by

$$(x_1, z_1) \cdot_L (x_2, z_2) = x_1x_2 - z_1z_2,\ ||(x, z)||_L^2 = (x, z) \cdot_L (x, z) = x^2 - z^2.$$

The hyperbolic trig functions parametrize $\mathcal{H}$: $x = \sinh s, z = \cosh s$.
When $s = 0$, this gives the point $(0, 1)$, when $s = \ln 2$ it gives $(\frac{3}{4}, \frac{5}{4})$.

$\mathcal{H}$ and S^1

The ambient space of $\mathcal{H}$ is Lorentz space, which is $\mathbb{R}^2$ equipped with the Lorentz pseudometric, in which the squared length of the vector (x, z) is $x^2 - z^2$. This is positive only if $|x| > |z|$. If P and Q are points of $\mathcal{H}$, then the vector $\vec{PQ}$ has positive squared length.

- $(\sinh s, \cosh s)$ is the point that is reached by travelling a distance s in the Lorentz pseudometric from $(1, 0)$ along $\mathcal{H}$ in the direction of increasing x. (In the other direction it's $(-\sinh s, \cosh s)$).
- $(\cos t, \sin t)$ is the point reached by travelling a distance t in the Euclidean metric counter-clockwise along S^1 from $(1, 0)$.

Hyperbolic Distance The (Lorentz) distance in $\mathcal{H}$ from $P(\sinh s_1, \cosh s_1)$ to $Q(\sinh s_2, \cosh s_2)$ is $|s_2 - s_1|$. In terms of the coordinates of P and Q, this is $\cosh^{-1}(-P \cdot_L Q)$ since

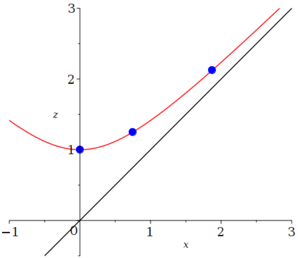
$$-P \cdot_L Q = \cosh s_1 \cosh s_2 - \sinh s_1 \sinh s_2 = \cosh(s_2 - s_1).$$

Hyperbolic Distance

For points P and Q of $\mathcal{H}$, $\text{distance}_L(P, Q) = \cosh^{-1}(-P \cdot_L Q)$.
For points R and S of S^1, their distance apart in S^1 is $\cos^{-1}(R \cdot S)$, where $\cdot$ is the ordinary (Euclidean) scalar product.

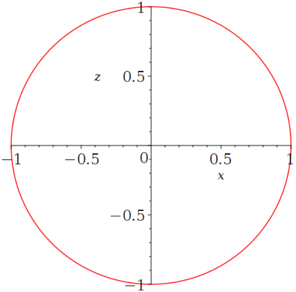
Warning Our picture of $\mathcal{H}$ does not represent hyperbolic distance accurately. The representation of $\mathcal{H}$ as the set of points in $\mathbb{R}^2$ satisfying $z^2 = x^2 + 1$ is not an isometric embedding of $\mathcal{H}$ in $\mathbb{R}^2$. Pairs of points that are the same distance apart in $\mathcal{H}$ do not appear so in the picture.

Example As s increases from $0(= \ln 1)$ to $\ln 2$, $(\sinh s, \cosh s)$ goes from $(0, 1)$ to $(\frac{3}{4}, \frac{5}{4})$. As s increases from $\ln 2$ to $2\ln 2 = \ln 4$, $(\sinh s, \cosh s)$ goes from $(\frac{3}{4}, \frac{5}{4})$ to $(\frac{15}{8}, \frac{17}{8})$. Both of these arcs of $\mathcal{H}$ have (hyperbolic) length equal to $\ln 2$.



Isometries of S^1

Definition An isometry of S^1 is a function from S^1 to S^1 that preserves distance along arcs. This means: if a, b are points of S^1 with images a', b', then $\text{distance}_{S^1}(a,b) = \text{distance}_{S^1}(a',b')$. Equivalently $a \cdot b = a' \cdot b'$, if points of S^1 are considered as unit vectors in $\mathbb{R}^2$.



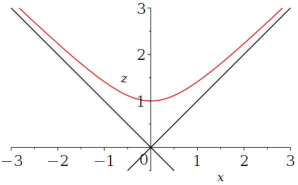
There are two types of isometries of S^1: rotations about the origin through any angle, and reflections in any diameter. Both arise from linear transformations of $\mathbb{R}^2$ with standard matrices as follows:

- Rotation R_α: $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $(\cos t, \sin t) \to (\cos(t+\alpha), \sin(t+\alpha))$
- Reflection M_α in the diameter through $(\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2})$:
  $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$, $(\cos t, \sin t) \to (\cos(\alpha - t), \sin(\alpha - t))$

All these matrices comprise the orthogonal group $O(2)$.

Isometries of $\mathcal{H}$

Definition An isometry of $\mathcal{H}$ is a function from S^1 to S^1 that preserves Lorentz distance along arcs. This means: if a, b are points of S^1 with images a', b', then $\text{distance}_{\mathcal{H}}(a, b) = \text{distance}_{\mathcal{H}}(a', b')$. Equivalently $a \cdot_L b = a' \cdot_L b'$.



One isometry of $\mathcal{H}$ is the Lorentz translation T_α, which maps $(\sinh s, \cosh s)$ to $(\sinh(s+\alpha), \cosh(s+\alpha))$ for a fixed $\alpha \in \mathbb{R}$.

The distance in $\mathcal{H}$ between $(\sinh s_1, \cosh s_1)$ and $(\sinh s_2, \cosh s_2)$ is $|s_1 - s_2|$, the same as $|(s_1+\alpha) - (s_2+\alpha)|$.

The T_α are analogous to rotations in S^1, and also arise from linear transformations of $\mathbb{R}^2$.

The standard matrix of $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is $\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$

Isometries of $\mathcal{H}$

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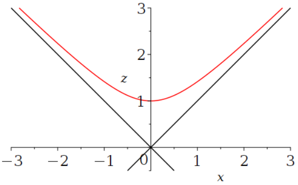

The reflection M_0 in $(0, 1)$ mapping $\mathcal{H} \to \mathcal{H}$ via $(\sinh s, \cosh s) \to (-\sinh s, \cosh s)$.

This restricts the reflection of $\mathbb{R}^2$ in the z-axis, with matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

For $\alpha \in \mathbb{R}$, the reflection M_α of $\mathcal{H}$ fixes $(\sinh \frac{\alpha}{2}, \cosh \frac{\alpha}{2})$ and maps

$$(\sinh s, \cosh s) \to (\sinh(\alpha - s), \cosh(\alpha - s)) .$$

M_α preserves distance in $\mathcal{H}$ and has matrix $\begin{pmatrix} -\cosh \alpha & \sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix}$.

Linear Algebra and/or Group Theory Interpretation

The set of matrices T_α and M_α describing isometries of $\mathcal{H}$ also forms a group, known as the Lorentz group. For a pair of vectors $u = \binom{a}{b}$ and $v = \binom{c}{d}$ in $\mathbb{R}^2$ their ordinary and Lorentz inner products are respectively given by the matrix products

- $u \cdot v \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = u^T I_2 v = u^T v.$
- $u \cdot_L v \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{L_2} \begin{pmatrix} c \\ d \end{pmatrix} = u^T L_2 v.$

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the ordinary scalar product. Then for any vectors $u, v \in \mathbb{R}^2$, $T(u) \cdot T(v) = u \cdot v$.
If M_T is the matrix of T, this means for all u and v that

$$(Mu)^T I_2 (Mv) = u^T I_2 v \Longrightarrow u^T M^T M v = u^T v,$$

for all $u, v \in \mathbb{R}^2$. This mean $M^T M = I_2$, so $M^{-1} = M^T$.