The normal vector and tangent plane to a surface

We'd like to extend the concept of curvature from curves to (smooth) surfaces and then assert that the plane has (constant) curvature 0, and that the unit sphere has (constant) curvature $+1$.

Let S be a smooth surface in $\mathbb{R}^3$ (no corners).

Let Q be a point in S.

There are many smooth curves in S through Q.
Somehow the curvature of the surface at Q will be defined in terms of their curvatures.

Every curve in S through P has a tangent vector at Q.

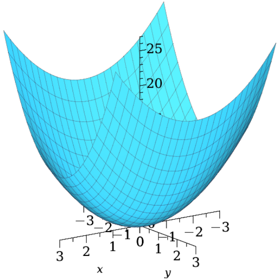
Claim All of these tangent vectors at Q to curves in S lie in a plane, called the tangent plane to S at Q.

Being a plane in $\mathbb{R}^3$, the tangent plane to S at Q has a normal direction.

Example: $S : z = x^2 + 2y^2$ (a paraboloid)

Find the equation of the tangent plane to the paraboloid S at the point $Q : (1, 2, 9)$.

Write $f(x, y, z) = x^2 + 2y^2 - z$.
$S : f(x, y, z) = 0$.
Gradient of f:
$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 4y, -1)$.
$\nabla(f)|_{(1,2,9)} = (2, 8, -1)$. This is the normal vector to P.



Equation of P: $\boxed{(2, 8, -1) \cdot (x - 1, y - 2, z - 9) = 0, 2x + 8y - z = 9.}$

If a smooth surface S has equation $f(x, y, z) =$constant, the equation of the tangent plane P to S at (x_0, y_0, z_0) is

$$P : \nabla f \cdot (x - x_0, y - y_0, z - z_0) = 0$$

What happened there?

Let S be a smooth surface in $\mathbb{R}^3$, with equation $f(x, y, z) =$constant.

Let $\mathcal{C}$ be a curve in S, parametrized by $t \to (f_1(t), f_2(t), f_3(t))$.

The tangent vector to $\mathcal{C}$ is $T = (f_1'(t), f_2'(t), f_3'(t))$.

On $\mathcal{C}$, f is a function of t via the dependence of x, y, z on t, and it is constant. Hence (on $\mathcal{C}$)

$$\boxed{0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot T.}$$

This is saying that at any point $Q = (x_0, y_0, z_0)$ where S is smooth, ∇f (at Q) is orthogonal to the tangent vector at Q of every curve in S that passes through Q. So ∇f is a normal vector to the tangent plane $T_Q(S)$ of S at Q.

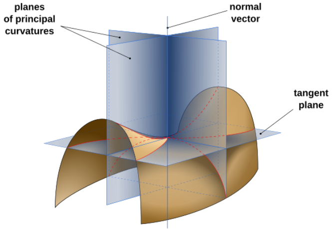
Definition ∇f (at Q) is called a normal vector to the surface S at the point Q. A unit normal vector is a unit vector in the direction of ∇f.

Normal Sections

On a smooth surface S, the (unit) normal vector at a point may have either of two opposite directions. We choose one of these (at a particular point) to be the unit normal vector, and extend the designation by requiring that the unit normal vector should vary continuously as we travel around the surface (so it doesn't abruptly reverse direction).

This amounts to choosing one "side" of the surface for the normal vector to point into, and it works provided that S is orientable - it fails on a Möbius band which is non-orientable.

At a point Q of S, let $\mathbf{n}$ be the unit normal.
Let L be the line through Q in the direction of $\mathbf{n}$.
Every plane in $\mathbb{R}^3$ that contains L intersects S in a curve (or a point).

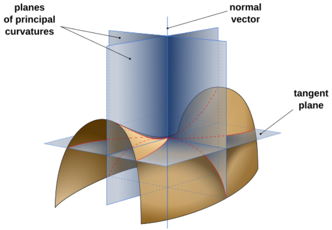


picture courtesy of Wikipedia

These curves are the normal sections of S at Q.

Let P be any plane in $\mathbb{R}^3$ containing L.
Rotating P about L gives all such planes,
and all normal sections of S at Q.

In a neighbourhood of Q, each normal section either lies on the side of the tangent plane into which $\mathbf{n}$ is directed, or the other side. For one side or the other, the curvature of a normal section is given a negative sign.



picture courtesy of Wikipedia

The normal curvatures of S at Q are the (signed) curvatures of the normal sections, as curves in R^3. The principal curvatures κ_1 and κ_2 of S at Q are the maximum and minimum normal curvatures.

Definition The Gaussian curvature κ of S at Q is the product $\kappa_1\kappa_2$.

Gaussian Curvature of S^2

Definition The Gaussian curvature κ of S at Q is the product $\kappa_1\kappa_2$.

- κ is positive at Q if all normal curvatures at Q have the same sign (all positive or all negative). This means that Q is a "cap" or a "cup" in S.
- κ is negative at Q if S has normal sections with both positive and negative curvatures at Q. This means that S is a saddle point.
- κ is zero at Q is all the non-zero normal curvatures have the same sign, and one of the principal curvatures is zero.

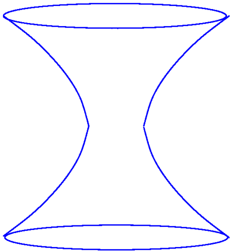
Example: The unit sphere $S^2 : x^2 + y^2 + z^2 = 1$.
A normal vector at the point Q (x_0, y_0, z_0) is (x_0, y_0, z_0). The normal sections are intersections of S^2 with planes containing the line OQ. They are great circles in S^2 - circles of radius 1, and curvature 1 in $\mathbb{R}^3$. So $\kappa_1 = \kappa_2 = 1$ (or both -1 depending on the direction of $\mathbf{n}$). Thus $\kappa_1\kappa_2 = 1$ at every point Q of S^2.

$\boxed{S^2 \text{ is a surface of constant Gaussian curvature 1.}}$

Surfaces of Constant Gaussian Curvature

- A sphere of radius R has constant Gaussian curvature $\frac{1}{R^2}$.
- A plane or cylinder has constant Gaussian curvature 0 (and a cylinder can be unrolled to a plane).
- What about a surface of constant Gaussian curvature -1?

Remark It is not too hard to visualize a surface where the Gaussian curvature is negative everywhere, like the picture here. For example take the surface of revolution obtained by rotating the hyperbola $y = \frac{1}{x}$ about the line $y = x$.



Surfaces like this have negative curvature everywhere, but not constant.