Euclidean and non-Euclidean Geometry (MA3101)
Lecture 6: The Spherical Cosine Rule

Dr Rachel Quinlan

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Triangles in the plane and sphere

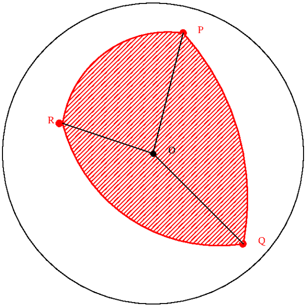
For a triangle in the plane with sides of lengths a and b, and an angle C between them, the length c of the third side is determined by the cosine rule.

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

In the special case where C is a right angle, $\cos C = 0$ and this says $c^2 = a^2 + b^2$.

A spherical triangle has three distinct points P, Q, R of S^2 as vertices, and its edges are segments of the great circles PQ, PR and QR. The angle at the vertex P is the angle of rotation of the sphere about the axis OP that moves PR to QR (through the triangle).

It is the dihedral angle between the planes OPR and OPQ.



The spherical cosine rule

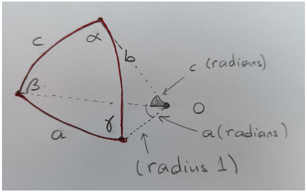
In a spherical triangle, it is still true that the (arc) length of two sides, and the angle between them, should determine the length of the third side. But spherical triangles don't satisfy the planar cosine rule. Here is the analogue.

Theorem 1

In a unit (radius 1) sphere, a triangle with side lengths a, b, c respectively opposite angles α, β, γ,

$$\cos a = \cos\alpha \sin b \sin c + \cos b \cos c$$

Key point about the relationship between arc lengths and angles: the angle a (in radians) is subtended at the centre of the sphere by the arc of length a. If the two vertices on this edge are P and Q, $\cos a$ is the scalar product $OP \cdot OQ$.

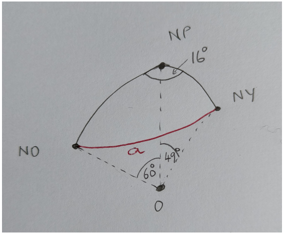


Application (MA3101 Summer 2022)

Problem Use the spherical cosine rule in the geodesic triangle whose vertices are New York (41° N, 74° W), New Orleans (30° N, 90° W) and the North Pole, to find the distance in km from New Orleans to New York. Take the Earth to be a sphere of radius 6500km.

$$\cos a = \cos\alpha \sin b \sin c + \cos b \cos c$$

Solution Temporarily think of the radius as "1 unit". From the latitudes we know the angles subtended at the centre of the Earth by the arcs $(NP)(NO)$ and $(NP)(NY)$. From the longitudes we know the spherical angle α at the North Pole (NP) (in degrees).



$$\cos a = cos(16^\circ)\sin(49^\circ)\sin(60^\circ) + \cos(49^\circ)\cos(60^\circ) \approx 0.9563.$$

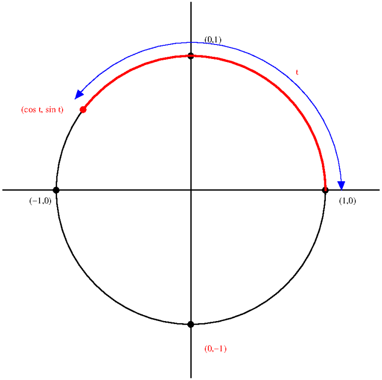
The arc length a is $\cos^{-1}(0.9563)$ in radians: ~ 0.2977.
Finally scale by the Earth's radius in km for an answer in km:
$0.2977 \times 6500 \approx$ **1935km**.

Parametric description of a (great) circle

In $\mathbb{R}^2$, the unit circle consists of all points with coordinates $(\cos t, \sin t)$, $t \in \mathbb{R}$.

The parameter t measures arc length counterclockwise from $(1, 0)$.

Thinking of $(1, 0)$ and $(0, 1)$ ($\vec{\imath}$ and $\vec{\jmath}$) as being "basis vectors", the unit circle in $\mathbb{R}^2$ consists of all points of the form



$$\cos t(1, 0) + \sin t(0, 1)$$

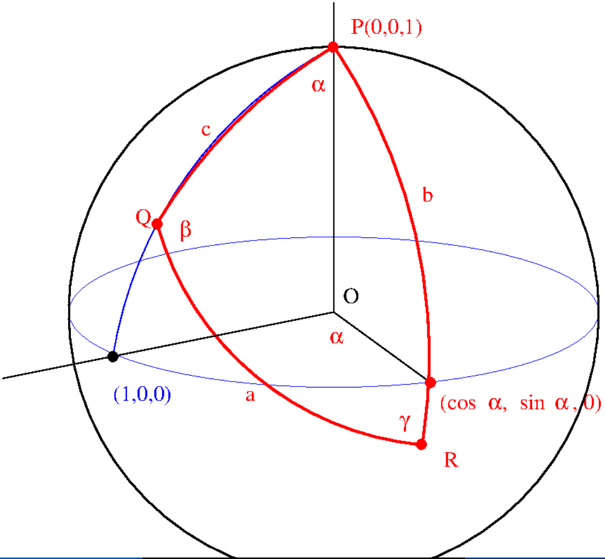
We can give a parametric description of any circle, for example in $\mathbb{R}^3$. If u and v are orthogonal vectors of the same length r in $\mathbb{R}^3$, then the circle through u and v with centre O is the set of all points of the form

$$\cos(t)\, u + \sin(t)\, v, \ \ t \in \mathbb{R}.$$

(We could take $t \in [0, 2\pi)$ instead of $t \in \mathbb{R}$ if we like.)

Proof of the spherical cosine rule

$$\cos a = \cos\alpha \sin b \sin c + \cos b \cos c$$



Proof of the spherical cosine rule

$$\boxed{\cos a = \cos\alpha \sin b \sin c + \cos b \cos c}$$

Set up the axes so that the vertex P with angle α is at $(0, 0, 1)$, and the vertex Q (with angle β) is in the XZ-plane.
Then the coordinates of Q are given by

$$\cos c(0, 0, 1) + \sin c(1, 0, 0) = (\sin c, 0, \cos c).$$

Proof of the spherical cosine rule

$$\boxed{\cos a = \cos\alpha \sin b \sin c + \cos b \cos c}$$

Set up the axes so that the vertex P with angle α is at $(0,0,1)$, and the vertex Q is at $(\sin c, 0, \cos c)$.
The third vertex is R and

$$\cos a = Q \cdot R.$$

The great circles PQ and PR respectively intersect the XY-plane at the points $(1,0,0)$ and $(\cos\alpha, \sin\alpha, 0)$.
From the parametric representation of the circle PR, the coordinates of R are given by

$$\cos b(0,0,1) + \sin b(\cos\alpha, \sin\alpha, 0) = (\sin b\cos\alpha, \sin b\sin\alpha, \cos b).$$

Finally

$$\cos a = \underbrace{(\sin c, 0, \cos c)\cdot(\sin b\cos\alpha, \sin b\sin\alpha, \cos b)}_{Q\cdot R} = \cos\alpha\sin b\sin c + \cos b\cos c.$$

Consequences of the spherical cosine rule

$$\boxed{\cos a = \cos\alpha \sin b \sin c + \cos b \cos c}$$

1. The Spherical Triangle Inequality In a spherical triangle PQR whose arc lengths a, b, c are all at most π, $a \leq b + c$, and $a = b + c$ only if all three vertices are on the same great circle.
   Proof Use the trigonometric identity
   $$\cos(b + c) = \cos b \cos c - \sin b \sin c.$$
   Since $\sin b$ and $\sin c$ are positive for $b, c \in (0, \pi)$, this means
   $$\cos a \geq \cos(b + c),$$
   and $\cos a = \cos(b + c)$ only if $\alpha = \pi$ which is the case of "spherical collinearity". Since $a \in [0, \pi]$ and the cosine function is strictly decreasing on this interval, it follows that $a \leq b + c$.
2. The Spherical Pythagorean Theorem If $\alpha = \frac{\pi}{2}$ then $\cos\alpha = 0$ and
   $$\cos a = \cos b \cos c.$$