Euclidean and non-Euclidean Geometry (MA3101)
Lecture 4: Euclid's Postulates and Spherical Geometry

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Euclid's five postulates

In the thirteen books of Euclid's Elements (c. 300BC), the following five postulates[1] are proposed as "the rules of the game" for geometry in the plane.

1. Any two distinct points are connected by a unique line segment.
2. A line segment can be extended indefinitely to a line.
3. A circle can be drawn with any centre and any radius.
4. All right angles are equal to each other.
5. Parallel Postulate (Playfair's version, 1795) Given a line L and a point P not on L, there is exactly one line through P that is parallel to L.

Euclidean geometry is the study of geometric systems where these axioms hold (although they do not form a complete axiomatic foundation).

Central to Euclid's postulates are the concepts of distance ("radius") and parallel lines.

Non-Euclidean geometries deviate from this framework, often by dropping the requirement for a concept of distance and/or by adapting the parallel postulate.

Non-Euclidean Geometries

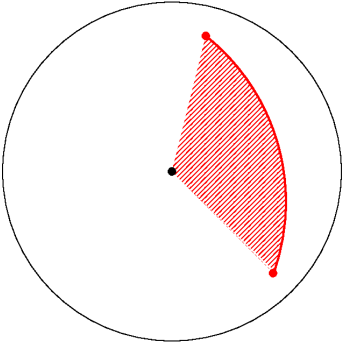
- Dropping the distance concept leads (for example) to projective geometry. This is the geometry of perspective, of projections of a 3-dimensional scene on a two-dimensional window.
- We get hyperbolic geometry by adapting the parallel postulate to: Given a line L and a point P not on L, there are at least two lines through P that do not intersect L.
- We get spherical geometry (or more generally elliptical geometry) by adapting the parallel postulate as follows: all pairs of lines intersect in at least one point.
  The sphere S^2 will be our model for this kind of geometry.

This lecture will look at some similarities and differences between S^2 and the Euclidean plane $\mathbb{E}^2$.

Some details to recall about circles and circular arcs

1. The circumference of the unit circle S^1 is 2π
   What does this mean? How do we define the length of a curve? We can think of the circumference of a circle as being the limit as $n \to \infty$ of the perimeter of a regular n-gon inscribed in the unit circle. This perimeter is $2n \times \sin\theta$, where $2n\theta$ is a full rotation. This limit exists and its value is (what we call) 2π.
2. Arc length
   The length of an arc on the unit circle is determined by the angle that is subtends at the centre. One radian is the angle at the centre of a circle that subtends an arc whose length is equal to the radius. The "full rotation" is 2π radians.
3. Every great circle (line) in S^2 is a copy of the unit circle.

Length of a line segment in S^2



For distinct (non-antipodal) points P and Q of S^2, what is the length of the (shorter) great circle arc ("line segment") joining them?

It is the angle α subtended by this arc at the origin (the centre).
The points P and Q are vectors of length 1 in $\mathbb{R}^3$.
Their scalar product $P.Q$ is $1 \times 1 \times \cos\alpha = \cos\alpha$. The arc length is

$$\boxed{d(P, Q) := \cos^{-1}(P.Q)}$$

Example: If P is $(0, 0, 1)$ and Q is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, then $P.Q = 0$ and $d(P, Q) = \cos^{-1}(0) = \pi/2$.

Triangles in the plane and sphere

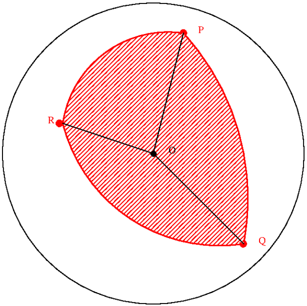
For a triangle in the plane with sides of lengths a and b, and an angle C between them, the length c of the third side is determined by the cosine rule.

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

In the special case where C is a right angle, $\cos C = 0$ and this says $c^2 = a^2 + b^2$.

A spherical triangle has three distinct points P, Q, R of S^2 as vertices, and its edges are segments of the great circles PQ, PR and QR. The angle at the vertex P is the angle of rotation of the sphere about the axis OP that moves PR to QR (through the triangle).

It is the dihedral angle between the planes OPR and OPQ.



The spherical cosine rule

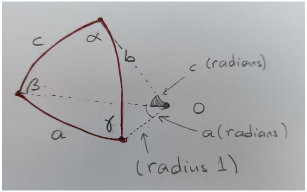
In a spherical triangle, it is still true that the (arc) length of two sides, and the angle between them, should determine the length of the third side. But spherical triangles don't satisfy the planar cosine rule. Here is the analogue.

Theorem 1

In a unit (radius 1) sphere, a triangle with side lengths a, b, c respectively opposite angles α, β, γ,

$$\cos a = \cos\alpha \sin b \sin c + \cos b \cos c$$

Key point about the relationship between arc lengths and angles: the angle a (in radians) is subtended at the centre of the sphere by the arc of length a. If the two vertices on this edge are P and Q, $\cos a$ is the scalar product $OP \cdot OQ$.

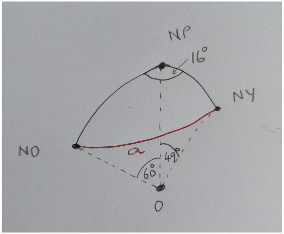


Application (MA3101 Summer 2022)

Problem Use the spherical cosine rule in the geodesic triangle whose vertices are New York (41° N, 74° W), New Orleans (30° N, 90° W) and the North Pole, to find the distance in km from New Orleans to New York. Take the Earth to be a sphere of radius 6500km.

$$\cos a = \cos\alpha \sin b \sin c + \cos b \cos c$$

Solution Temporarily think of the radius as "1 unit". From the latitudes we know the angles subtended at the centre of the Earth by the arcs $(NP)(NO)$ and $(NP)(NY)$. From the longitudes we know the spherical angle α at the North Pole (NP) (in degrees).



$$\cos a = \cos(16^\circ)\sin(49^\circ)\sin(60^\circ) + \cos(49^\circ)\cos(60^\circ) \approx 0.9563.$$

The arc length a is $\cos^{-1}(0.9563)$ in radians: ~ 0.2977.
Finally scale by the Earth's radius in km for an answer in km:
$0.2977 \times 6500 \approx$ **1935km**.