Euclidean and non-Euclidean Geometry (MA3101)
Lecture 4: Geometry - Measuring the Earth

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Geometry means “Measuring the Earth”

from a 14th century translation of Euclid’s *Elements*

Geometry means “measuring the earth”.

Its early history was motivated by practicalities of measurement, travel, construction and navigation, but led to mathematical insights that are still current. We’ll look at a couple of examples.

Rough Timeline

1800BC Volume of truncated square pyramid (Thebes, “Moscow” Papyrus)

1750BC Written statement of the Theorem of Pythagoras (Tell al-Dhabbai, Iraq)

1650BC $\pi \sim 3.16$ (Thebes, Ahmes (“Rhind”) Papyrus)

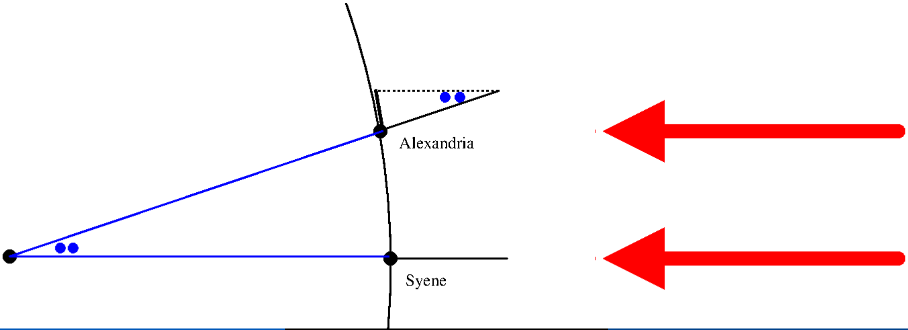
300BC Euclid’s Elements

1637 Coordinate (analytic) geometry (La Géometrie, Descartes)

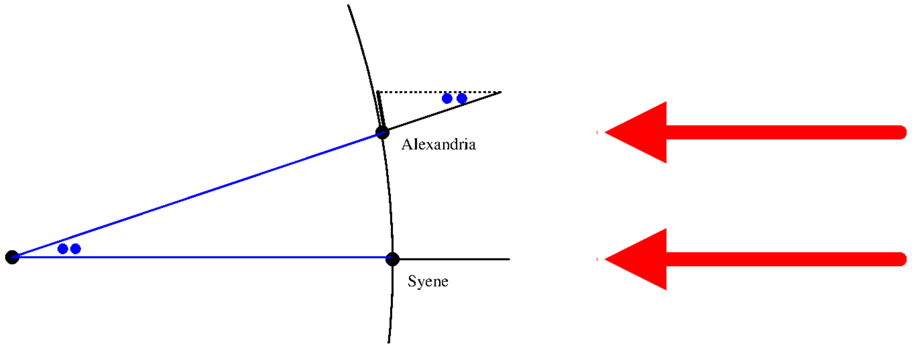
2006 Proof of Thurston’s *Geometrization Conjecture* by Perelman

Circumference of the Earth (Eratosthenes, ($\sim$250BC))

Syene (modern Aswan) is on the Tropic of Cancer.
The vertical sun at noon on the summer solstice shone straight down a well in Syene, casting no shadow.
Alexandria is north of Syene.
At the same time, a vertical pole in Alexandria cast a measurable shadow, showing an angle of (approx) 7° with the sun's rays - roughly $\frac{1}{50}$ of a full circle.

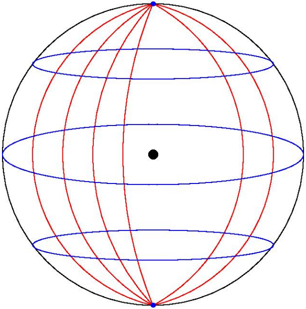


Circumference of the Earth (Eratosthenes, (∼250BC))



Conclusion since the two angles highlighted in blue are equal, the (known) distance from Syene to Alexandria should be $\frac{1}{50}$ of the great circle through both of them.
Estimated circumference; 39,060km to 40,320km.
"Correct" value is 40,008km for meridianal circumference, equatorial circumference is 40,075km.

Maps of the World

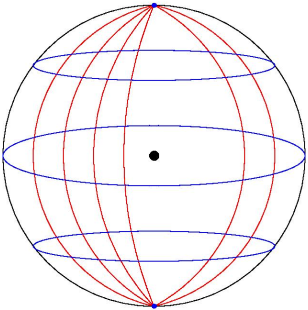


The plane and sphere are not isometric to each other. This is basically saying we can't wrap a piece of paper around a sphere, like we can around a cylinder.

A planar map of a spherical surface cannot accurately present (relative) distances, angles and areas: some distortion is unavoidable.

Methods for making maps of the world involve functions from the sphere to the plane. Choosing a suitable function means deciding what information is prioritized and what kind of distortion is acceptable.

Maps of the World



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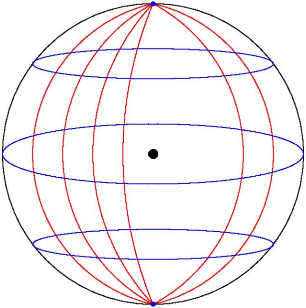
Parallels of latitude (blue in the picture) are not great circles. The length of the parallel at latitude ϕ (north or south) is $E \cos \phi$, where E is the length of the equator. The (north-south) distance between two parallels that are 1° apart is approximately 111km -this is the same all over the world.

Maps of the World



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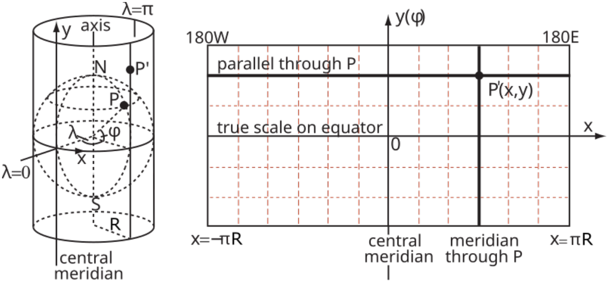
Meridians of longitude are great (semi)circles, and all have the same length. But the (east-west) distance between two meridians that are 1° apart depends on the latitude. The meridians all intersect at the poles.

Example: Cylindrical Projections[1]



The idea is to wrap a piece of paper into a cyclinder around the globe, that touches it at the equator. Each line of longitude is mapped (according to some function) to the vertical line on the paper where they touch at the equator. The poles are excluded and the lines of latitude and longitude are mapped to a rectangular grid. The parallel at latitude ϕ (north or south) is stretched by a factor $\frac{1}{\cos\phi}$. That's 1.67 in Galway (53° N) and 2.87 in Tromsø(70° N).

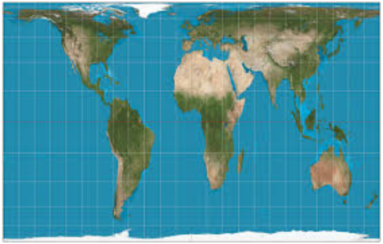
[1] pictures from Wikipedia

Example: Cylindrical Projections[1]

An example is the Mercator projection (1589), which is widely used in online maps and in marine charts. It portrays paths of constant direction (called rhumbs or loxodromes) as straight lines in the plane. But it vastly distorts distance and area far from the equator (for example it makes Greenland look bigger than the continent of Africa).

[1] pictures from Wikipedia

Example: Cylindrical Projections[1]



Another example is the Gall-Peters projection (1855 and 1973). It introduces a vertical distortion to compensate for the horizontal distortion in the Mercator representation, so that (relative) area is accurate (but shape and distance are not).

[1] pictures from Wikipedia

Example: Icosahedral Projection

Another approach (besides cylindrical projection) is to approximate the sphere with a polyhedron, which has faces that are polygons.

There are five convex regular polyhedra (the platonic solids) that are "as symmetric as possible": the tetrahedron, cube, octhedron, dodecahedron and icosahedron.

The icosahedron has the greatest number of faces (20). An icosahedral projection partitions the spherical surface into 20 regions, each mapped onto an equilateral triangle. The icosahedral map can be unfolded onto a plane.