Euclidean and non-Euclidean Geometry (MA3101)
Lecture 3: What does “straight” mean?

Dr Rachel Quinlan

September 17, 2024

What is a straight line? [2]

- In 2-dimensional coordinate space $\mathbb{R}^2$, it's the set of points (x, y) that satisfy an equation of the form $ax + by = c$ or a set of points of the type $\{(a, b) + t(c, d) : t \in \mathbb{R}\}$ (parametric description).
- For any pair of distinct points P and Q in $\mathbb{R}^2$ (or $\mathbb{R}^n$ generally), there is a unique line that contains both of them, so there is a unique line segment that joins them.
- The length of this line segment is the (Euclidean) distance between P and Q. There is no shorter path from P to Q in $\mathbb{R}^n$. [1]
- The Theorem of Pythagoras give us a way of computing Euclidean distance in terms of coordinates.

  In $\mathbb{R}^2$ and $\mathbb{R}^3$ we can define a line segment and a straight line in terms of (minimum) distance. But this does not extend so well to geometries where we can't calculate distance. How else can we describe a straight line?

[1] This seemingly harmless assertion relies on having a concept of length for paths that are not line segments, which takes some effort.

[2] This week's lectures are partly based on Chapters 1,2,4 of *Experiencing Geometry* by David Henderson.

What does a straight line look like?

It has some symmetries:

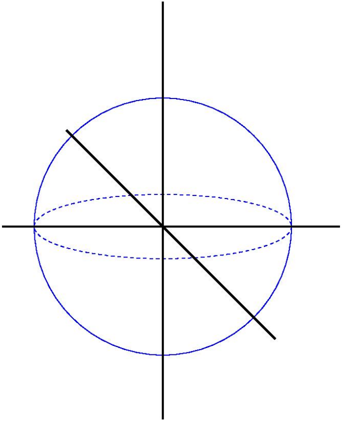
1. A line divides the plane into two parts that reflect on to each other across the line.
2. A line is mapped onto itself by a reflection in any line perpendicular to it, or a 180° rotation about any of its points.
3. A line in $\mathbb{R}^3$ is preserved by rotation (through any angle) about itself as an axis.

Practicalities These observations (1. and 3. in particular) have practical applications in craft and manufacturing, in checking the straightness of edges or of manufactured components.

How do we construct a straight line? This is not easy! One way to do it is to fold a piece of paper (this uses 1. above). It is easier to construct a circle, that just requires a compass.

The sphere S^2

Definition The 2-sphere S^2 is the set of points in $\mathbb{R}^3$ that satisfy the equation $x^2 + y^2 + z^2 = 1$.



- S^2 is the set of points at distance 1 from O.
- S^2 consists only of these points, not the interior of the ball that they enclose.
- Recall the "right-hand" convention for the axis orientations in $\mathbb{R}^3$: if the first finger and middle finger of a right hand point along the positive X and Y-axes, the thumb points along the positive Z-axis.

What should we call a straight trajectory on the surface of a sphere, like the one we live on?

What is a straight line on the surface of a sphere?

What is a straight line on the surface of a sphere?

Is this blue line on Google maps "straight"?

Great circles

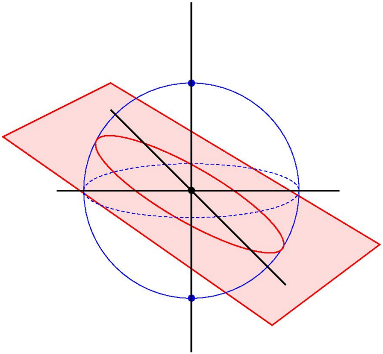
Definition A great circle in S^2 is the intersection of S^2 with any plane through the origin in $\mathbb{R}^3$.

A great circle on any sphere is an "equator", or a circle on the sphere whose centre and radius coincide with that of the sphere.

On the sphere, great circles are geodesics. This means that they are straight in the intrinsic geometry of the sphere.

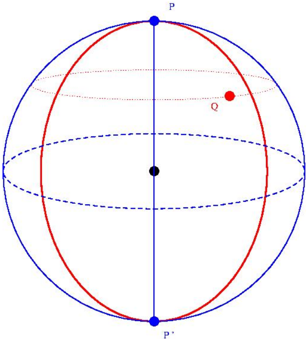
- The local environment has a reflection symmetry in the geodesic.
- An inhabitant of the surface of the sphere, whose awareness is confined to the surface, senses a geodesic as "straight" in terms of symmetry.
- Extrinsically, there is a Euclidean line in $\mathbb{R}^3$ from Lusaka to La Paz through the earth's interior. As inhabitants of the surface we do not see this as a route, and we see the "direct" route as the geodesic (great circle) segment on the earth's surface.

Features of Spherical Geometry



Given a pair of distinct points on a sphere, is there a always a line (i.e. great circle) that includes both of them?
When is there exactly one such line?
When are there multiple such lines?

How does this situation compare to lines in the plane with Euclidean geometry?
Is there a meaning of “parallel” for great circles?
When are three points on the sphere (intrinsically) collinear?
Do we have some (versions of familiar) theorems about triangles on a sphere?



Let P and Q be distinct and non-antipodal (i.e. not "opposite") points on the sphere. Let P' be the antipode of P. Let G be any great circle that includes P and P' (red in the picture). Rotating this through any angle about the axis PP' gives another great circle including P and P'. One of these contains Q.

Theorem The shortest path on the sphere from P to Q is along the great circle that contains P and Q.
"Proper" proof later, for now we stick with "proof by piece of string".

Remark Given any point P on a sphere, there is a unique great circle which is the "equator" with respect to the "poles" P and and its antipode P'. In this way we have a correspondence between great circles and pairs of antipodal points.

Geodesics on other surfaces

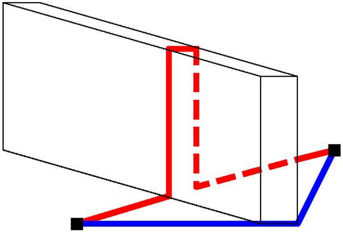
We can use the "ribbon test" to physically identify geodesics ("straight" lines or paths) on other surfaces, such as the surface of a cylinder or a cone.

If a cylindrical or conical surface is folded smoothly from a sheet of paper, a geodesic on the surface looks like a straight line when the paper is laid flat.

On the sphere, every great circle is the intersection of the sphere with a plane that includes its centre. We can consider whether or when the intersection of a plane with a cylinder or cone is a geodesic. A quick way to explore this is to fold a cylinder or cone from paper, dip it in a bowl of water, then unfold and look at the water line.

On the (crocheted) hyperbolic plane (more on this later!) you can check for a geodesic between any two points by gently pulling the two points apart and looking at the symmetry along a curve between them, where the surface starts to fold.

Does “shortest” always mean “straight”?



Not in this (not-in-perspective) picture, where the “straight” path is the red one (think of a ribbon), but the blue path is shorter. The surface in this picture is not smooth, it has corners and edges.

(Vague) Theorem If a surface is smooth then a geodesic on the surface is always the shortest path between “nearby” points. If the surface is complete (every geodesic segment can be extended indefinitely), then every pair of points is connected by a geodesic that is the shortest path between them.

For an interpretation of “nearby”, think of a helix on a cylinder.
An example of a “non-complete” surface would be a plane with a hole cut out.