Rings (MA 416) 2024-2025

Assignment 1

Due date: Wednesday October 9

Problems marked with $*$ are for submission. The rest are for independent study and discussion in the tutorials. Please submit solutions via the Canvas page, or in person at the lecture if you prefer paper.

1. Decide whether each of the following objects is a ring. In each case if you think the object in question *is* a ring, it is sufficient to just say so; if not you should give at least one reason why not.

   (a) The set $2\mathbb{Z}$ of even integers, with the usual addition and multiplication of integers.

   (b) The set $M_2(\mathbb{Z})$ consisting of all 2×2 matrices with integer entries, with matrix addition and matrix multiplication.

   (c) The set of complex numbers of the form $a+bi$, where $a, b \in \mathbb{Z}$, with the usual addition and multiplication of complex numbers.

   (d) The subset of $\mathbb{C}$ consisting of those complex numbers with real part equal to zero (e.g. $2i, \sqrt{3}i, -i$ etc.), under the usual addition and multiplication of complex numbers.

   (e) The set of real numbers of the form $a + b\sqrt{2}$, where a and b are rational numbers (under the addition and multiplication of real numbers).

   (f) * The set of real numbers of the form $a + b\sqrt[3]{2}$, where a and b are rational numbers (under the addition and multiplication of real numbers, where $\sqrt[3]{2}$ denotes the real cube root of 2).

   (g) The set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with addition defined as in Lecture 1 but with multiplication defined differently : for continuous functions f and g define the product of f and g to be the composition $f \circ g$; i.e.

   $$f \circ g(x) = f(g(x)) \text{ for } x \in \mathbb{R}.$$

   (h) * The set of polynomials in $\mathbb{Q}[X]$ in which the constant term is an integer.

2. * Define a binary operation $\star$ on the set $2\mathbb{Z}$ of even integers by

   $$2k \star 2m = 2km,$$

   for even integers $2k$ and $2m$. (So for example $6 \star 10 = 30$.)
   Determine (with explanation) whether $2\mathbb{Z}$ is a ring, with the usual addition operation and with $\star$ as multiplication.

3. Define a (different) "multiplication" $\square$ of even integers by

   $$2k\square 2m = km.$$

   So for example $6\square 10 = 15$.
   Is the set of even integers a ring, with the usual addition and with $\square$ as multiplication?

4. List five different examples of non–commutative rings, including at least one example of a non–commutative ring containing a finite number of elements.

5. (a) Write out the multiplication tables for the sets of non–zero elements of $\mathbb{Z}/7\mathbb{Z}$ and $\mathbb{Z}/8Z$. List the units of each of these rings.
   (b) Prove that the non–zero element $\bar{a}$ of $\mathbb{Z}/n\mathbb{Z}$ is a unit in $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(a, n) = 1$. Deduce that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is prime.
   (c) Write out the multiplication table of the group $\mathcal{U}(\mathbb{Z}/12\mathbb{Z})$.
   (d)* Prove that the non–zero element $\bar{a}$ of $\mathbb{Z}/n\mathbb{Z}$ is a zero–divisor in $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(a, n) > 1$. Deduce that every non–zero element of $\mathbb{Z}/n\mathbb{Z}$ is either a zero–divisor or a unit.

6. (a) * Show that the unit group of $\mathbb{Z}/7\mathbb{Z}$ is a cyclic group of order 6.
   (b) Show that the unit group of $\mathbb{Z}/8\mathbb{Z}$ is not cyclic.

7. *
   (a) What is the unit group of the ring $M_2(\mathbb{Z})$ of 2×2 matrices whose entries are integers? Explain your answer.
   (b) What is the unit group of the polynomial ring $\mathbb{Q}[X]$?

8. Let R be a ring with identity element 1_R for multiplication, and with zero element 0_R. The *characteristic* of R is defined to be the least positive integer m for which

$$\underbrace{1_R + 1_R + \cdots + 1_R}_{m \text{ copies of } 1_R} = 0_R,$$

if such an integer exists. If there is no such m, then R has characteristic zero (for example $\mathbb{Z}$ has characteristic zero).
   (a) Give an example of a finite commutative ring of characteristic 5.
   (b) Give an example of a finite non-commutative ring of characteristic 5.
   (c) Give an example of an infinite commutative ring of characteristic 5.
   (d) * Give an example of an infinite non-commutative ring of characteristic 5.

9. *
   (a) Show that $(x + y)^2 = x^2 + y^2$ for all elements x and y of a commutative ring of characteristic 2.
   (b) Show that $(x + y)^3 = x^3 + y^3$ for all elements x and y of a commutative ring of characteristic 3.
   (c) Give an example to show that $(x+y)^4$ is not necessarily equal to x^4+y^4, for elements x and y of a commutative ring of charactersitic 4.
   (d) Give an example to show that $(x+y)^2$ is not necessarily equal to x^2+y^2, for elements x and y of a (non-commutative) ring of characteristic 2.

10. * Identify all binary operations $\star$ that can be defined on $\mathbb{Z}$, for which $\mathbb{Z}$ is a ring with the usual addition and with $\star$ as multiplication.
(The usual multiplication operation on $\mathbb{Z}$ is one candidate for $\star$, the question is asking if there are other possibilities for a multiplication operation and what they are if so.)