## Upper and Lower Bounds

## Definition 46

Let $S$ be a subset of $\mathbb{R}$. An element $b$ of $\mathbb{R}$ is an upper bound for $S$ if $x \leq b$ for all $x \in S$. An element $a$ of $\mathbb{R}$ is a lower bound for $S$ if $a \leq x$ for all $x \in S$.

Recall that
■ $S$ is bounded above if it has an upper bound,

- $S$ is bounded below if it has a lower bound,
- $S$ is bounded if it is bounded both above and below.

In this section we are mostly interested in sets that are bounded on at least one side.

## Maximum and minimum elements

## Definition 47

Let $S$ be a subset of $\mathbb{R}$. If there is a number $m$ that is both an element of $S$ and an upper bound for $S$, then $m$ is called the maximum element of $S$ and denoted $\max (S)$.
If there is a number / that is both an element of $S$ and a lower bound for $S$, then I is called the minimum element of $S$ and denoted by $\min (S)$.

## Notes

A set can have at most one maximum (or minimum) element.

## Maximum and minimum elements

## Definition 47

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## Notes

Pictorially, on the number line, the maximum element of $S$ is the rightmost point that belongs to $S$, if such a point exists. The minimum element of $S$ is the leftmost point on the number line that belongs to $S$, if such a point exists.

## Not every set has a maximum element

There are basically two reasons why a subset $S$ of $\mathbb{R}$ might fail to have a maximum element. First, $S$ might not be bounded above - then it certainly won't have a maximum element.

Secondly, $S$ might be bounded above, but might not contain an element that is an upper bound for itself. Take for example an open interval like $(0,1)$. This set is certainly bounded above. However, take any element $x$ of $(0,1)$. Then $x$ is a real number that is strictly greater than 0 and strictly less than 1 . Between $s$ and 1 there are more real numbers all of which belong to $(0,1)$ and are greater than $x$. So $x$ cannot be an upper bound for the interval $(0,1)$.


An open interval like $(0,1)$, although it is bounded, has no maximum element and no minimum element.
An example of a subset of $\mathbb{R}$ that does have a maximum and a minimum element is a closed interval like $[2,3]$. The minimum element of $[2,3]$ is 2 and the maximum element is 3 .

Remark : Every finite subset of $\mathbb{R}$ has a maximum element and a minimum element.

## Supremum and Infimum

For bounded subsets of $\mathbb{R}$, there are notions called the supremum and infimum that are closely related to maximum and minimum. Every subset of $\mathbb{R}$ that is bounded above has a supremum and every subset of $\mathbb{R}$ that is bounded below has an infimum.

## Definition 48 (The Axiom of Completeness for $\mathbb{R}$ )

Let $S$ be a subset of $\mathbb{R}$ that is bounded above. Then the set of all upper bounds for $S$ has a minimum element. This number is called the supremum of $S$ and denoted $\sup (S)$.
Let $S$ be a subset of $\mathbb{R}$ that is bounded below. Then the set of all lower bounds for $S$ has a maximum element. This number is called the infimum of $S$ and denoted $\inf (S)$.

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Let $S$ be a subset of $\mathbb{R}$ that is bounded below. Then the set of all lower bounds for $S$ has a maximum element. This number is called the infimum of $S$ and denoted $\inf (S)$.

Notes
1 The supremum of $S$ is also called the least upper bound (lub) of $S$.
2 The infimum of $S$ is also called the greatest lower bound ( $\mathrm{g} \mid \mathrm{b}$ ) of $S$.

The definition above is simultaneously a definition of the terms supremum and infimum and a statement of the Axiom of Completeness for the real numbers.

To see why this statement says something special about the real numbers, temporarily imagine that the only number system available to us is $\mathbb{Q}$, the set of rational numbers. Look at the set

$$
S:=\left\{x \in \mathbb{Q}: x^{2}<2\right\} .
$$

$S:=\left\{x \in \mathbb{Q}: x^{2}<2\right\}$

So $S$ consists of all those rational numbers whose square is less than 2 . It is bounded above, for example by 2.
The positive elements of $S$ are all those positive rational numbers that are less than the real number $\sqrt{2}$.

Claim: $S$ does not have a least upper bound in $\mathbb{Q}$.
To see this, suppose that $x$ is a rational number that is a candidate for being the least upper bound of $S$ in $\mathbb{R}$.

- If $x^{2}<2$, then there is a gap in the number line between $x$ and $\sqrt{2}$, and in this gap are rational numbers that are greater than $x$ but still less than $\sqrt{2}$. So $x$ is not an upper bound of $S$.
- If $x^{2}>2$, then there is a gap in the number line between $\sqrt{2}$ and $x$, and in this gap are rational numbers that are still upper bounds of $S$ but are less than $x$.

If we consider the same set $S$ as a subset of $\mathbb{R}$, we can see that $\sqrt{2}$ is the supremum of $S$ in $\mathbb{R}$ (and $-\sqrt{2}$ ) is the infimum of $S$ in $\mathbb{R}$.

This example demonstrates that the Axiom of Completeness does not hold for $\mathbb{Q}$, i.e. a bounded subset of $\mathbb{Q}$ need not have a supremum in $\mathbb{Q}$ or an infimum in $\mathbb{Q}$.

## Question 49

Let $S=\left\{\frac{2 n+4}{3 n}: n \in \mathbb{Z}, n \geq 1\right\}$.
1 List four elements of $S$.
2 Identify, with explanation, the maximum element of $S$.
3 Show that $S$ has no minimum element, and determine the infimum of $S$.

## Learning Outcomes for Section 2.5

After studying this section you should be able to

- State what it means for a subset of $\mathbb{R}$ to be bounded (or bounded above or bounded below).
■ Define the terms maximum, minimum, supremum and infimum and explain the connections and differences between them.
- State the Axiom of Completeness.
- Determine whether a set presented like the one in the problem above is bounded (above and/or below) or not and identify its maximum/minimum/infimum/supremum as appropriate, with explanation.


## Chapter 3: Sequences, series and convergence

Section 3.1: Introduction to sequences and series

## Question 50

Does it make sense to talk about the "number"

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2}}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots ?
$$

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- $1+\frac{1}{4}=1.25$

$$
\frac{\pi^{2}}{6} \approx 1.644934
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- $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16} \approx 1.423611$

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- $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(10)^{2}} \approx 1.549767$

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■ $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(200)^{2}} \approx 1.639947$

- $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(10000)^{2}} \approx 1.644834$

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$$

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■ $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(10000)^{2}} \approx 1.644834$

- $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(100000)^{2}} \approx 1.644924$

$$
\frac{\pi^{2}}{6} \approx 1.644934
$$

## The series $\sum_{n=1}^{\infty} \frac{1}{n^{2}}$

The series

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2}}
$$

converges to the number $\frac{\pi^{2}}{6}$ (we will have precise definitions for the highlighted terms a bit later).

## The series $\sum_{n=1}^{\infty} \frac{1}{n^{2}}$

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$$
\sum_{n=1}^{\infty} \frac{1}{n^{2}}
$$

converges to the number $\frac{\pi^{2}}{6}$ (we will have precise definitions for the highlighted terms a bit later).

This fact is remarkable - there is no obvious connection between $\pi$ and squares of the form $\frac{1}{n^{2}}$; moreover all the terms in the series are rational but $\frac{\pi^{2}}{6}$ is certainly not.
This example gives us in principle a way of calculating the digits of $\pi$ or at least of $\pi^{2}$. (In practice there are similar but better ways, as the convergence in this example is very slow).

## Another Example

## Example 51

What about

$$
\sum_{n=1}^{\infty} \frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots ?
$$

Try experimenting with initial segments again :

- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{50} \approx 4.4992$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

## Another Example

## Example 51

What about

$$
\sum_{n=1}^{\infty} \frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots ?
$$

Try experimenting with initial segments again :

- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{50} \approx 4.4992$
- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100} \approx 5.1874$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

## Another Example

## Example 51

What about

$$
\sum_{n=1}^{\infty} \frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots ?
$$

Try experimenting with initial segments again :

- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{50} \approx 4.4992$
- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100} \approx 5.1874$
- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000} \approx 7.4855$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

## Another Example

## Example 51

What about

$$
\sum_{n=1}^{\infty} \frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots ?
$$

Try experimenting with initial segments again :

- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{50} \approx 4.4992$
- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100} \approx 5.1874$
- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000} \approx 7.4855$
- $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{50000} \approx 11.3970$

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course.

## Another Example ...

## Example 52

What about

$$
\sum_{n=1}^{\infty} \frac{1}{2^{2 n}}=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots ?
$$

Experimenting reveals

- $\frac{1}{4}+\frac{1}{16}=\frac{5}{16}$
- $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}=\frac{341}{1024} \approx 0.33301$
- $\frac{1}{2^{2}}+\frac{1}{2^{4}}+\frac{1}{2^{6}}+\cdots+\frac{1}{2^{14}} \approx 0.3333$

These calculations can be verified directly using properties of sums of geometric progressions. It appears that this series is converging (quite fast) to $\frac{1}{3}$.

## Another Example

## Example 53

What about

$$
\sum_{n=1}^{\infty} \frac{1}{2^{2 n}}=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots ?
$$

The following picture gives some graphical evidence for this hypothesis.


## A last example

## Example 54

Does it make sense to talk about

$$
f(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\ldots
$$

as a function of $x$ ?
If it does, then $f$ must have a domain (consisting of some or all of the real numbers?) and substituting these values in to the definition in place of $x$ must somehow make sense.

- $x=0: f(0)=0$

In all cases we get (just from the first six terms) something very close to $\sin x$.

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- $x=0: f(0)=0$
- $x=\frac{\pi}{2}: f\left(\frac{\pi}{2}\right) \approx 0.9999$ (six terms)

In all cases we get (just from the first six terms) something very close to $\sin x$.

## A last example

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- $x=0: f(0)=0$
- $x=\frac{\pi}{2}: f\left(\frac{\pi}{2}\right) \approx 0.9999$ (six terms)
- $x=\frac{\pi}{6}: f\left(\frac{\pi}{6}\right) \approx 0.5000$ (six terms)

In all cases we get (just from the first six terms) something very close to $\sin x$.

## A last example

## Example 54

Does it make sense to talk about

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f(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\ldots
$$

as a function of $x$ ?
If it does, then $f$ must have a domain (consisting of some or all of the real numbers?) and substituting these values in to the definition in place of $x$ must somehow make sense.

- $x=0: f(0)=0$
- $x=\frac{\pi}{2}: f\left(\frac{\pi}{2}\right) \approx 0.9999$ (six terms)
- $x=\frac{\pi}{6}: f\left(\frac{\pi}{6}\right) \approx 0.5000$ (six terms)
- $x=\frac{\pi}{3}: f\left(\frac{\pi}{3}\right) \approx 0.8660$ (six terms) $\left(\frac{\sqrt{3}}{2} \approx 0.8660\right)$

In all cases we get (just from the first six terms) something very close to $\sin x$.

