An Example (from 2015 Summer Exam)

Example 67

A sequence (a_n) of real numbers is defined by

$$a_0 = 4, \;\; a_n = \frac{1}{2}(a_{n-1} - 2) \text{ for } n \geq 1.$$

1. *Write down the first four terms of the sequence.*
2. *Show that the sequence is bounded below.*
3. *Show that the sequence is montonically decreasing.*
4. *State why it can be deduced that the sequence is convergent, and determine its limit.*

Note: This is an example of a sequence that is defined recursively. This means that the first term is given and subsequent terms are defined (one by one) in terms of previous ones. We are not given a general formula for the nth term although one may exist.

Learning Outcomes for Section 3.2

After studying this section you should be able to

- Explain what a sequence is;
- State what it means for a sequence to be
  - convergent or divergent;
  - bounded or unbounded (above or below);
  - monotonic, increasing or decreasing.
- Give and/or identify examples of sequences with or without various properties (or combinations of properties) from the above list;
- State, prove and apply the Monotone Convergence Theorem;
- Analyze examples similar to Example 83.

Section 3.3: Introduction to Infinite Series

Definition 68

A series or infinite series is the sum of all the terms in a sequence.

Example 69 (Examples of infinite series)

1. $\displaystyle\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

Section 3.3: Introduction to Infinite Series

Definition 68

A series or infinite series is the sum of all the terms in a sequence.

Example 69 (Examples of infinite series)

1. $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$
2. *A geometric series*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

Every term in this series is obtained from the previous one by multiplying by the common ratio $\frac{1}{2}$. This is what geometric means.

Examples of Series (continued)

Example 70

3. *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

Examples of Series (continued)

Example 70

3. *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

4. *An alternating series*

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

Notes

1. For now these infinite sums are just formal expressions or arrangements of symbols. Whether it is meaningful to think of them as numbers or not is something that can be investigated.

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Notes

1. For now these infinite sums are just formal expressions or arrangements of symbols. Whether it is meaningful to think of them as numbers or not is something that can be investigated.
2. A series is not the same thing as a sequence - don't confuse these terms! A sequence is a list of numbers. A series is an infinite sum.
3. The "sigma" notation for sums: sigma (lower case σ, upper case Σ) is a letter from the Greek alphabet, the upper case $\sum$ is used to denote sums. The notation $\sum_{n=i}^{j} a_n$ means:
   i and j are integers and $i \leq j$. For each n from i to j the number a_n is defined; the expression above means the sum of the numbers a_n where n runs through all the values from i to j, i.e.

   $$\sum_{n=i}^{j} a_n = a_i + a_{i+1} + a_{i+2} + \cdots + a_{j-1} + a_j.$$

   For infinite sums we can have $-\infty$ and/or ∞ (instead of fixed integers i and j) as subscripts and superscripts for the summation.

Sequences of partial sums

In the examples above we can start from the beginning, adding terms at the start of the series. Adding term by term we get the following lists.

1. $\displaystyle\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

   $1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, \dots$ $1, 3, 6, 10, 15, \dots$

   Since the terms being added on at each stage are getting bigger, the numbers in the list above will keep growing (faster and faster as n increases) - we can't associate a numerical value with this infinite sum.

Examples (continued)

2. A geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3} \dots \qquad 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32} \dots$$

In this example the terms that are being added on at each step ($\frac{1}{2^n}$) are getting smaller and smaller as n increases, and the numbers in the list appear to be converging to 2.

Examples (continued)

2. A geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3} \dots \qquad 1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{31}{16},\ \frac{63}{32} \dots$$

In this example the terms that are being added on at each step ($\frac{1}{2^n}$) are getting smaller and smaller as n increases, and the numbers in the list appear to be converging to 2.

3. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \dots \qquad 1,\ \frac{3}{2},\ \frac{11}{6},\ \frac{25}{12},\ \frac{137}{60}, \dots$$

It is harder to see what is going on here.

4. An alternating series

$$\sum_{n=0}^{\infty}(-1)^n = 1 + (-1) + 1 + (-1) + \ldots$$

$1,\ 1-1,\ 1-1+1,\ 1-1+1-1,\ 1-1+1-1+1\ldots \quad 1, 0, 1, 0, 1, \ldots$

The terms being "added on" at each step are alternating between 1 and -1, and as we proceed with the summation the "running total" alternates between 0 and 1. There is no numerical value that we can associate with the infinite sum $\sum_{n=0}^{\infty}(-1)^n$.

4. An alternating series

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Note: The series in 2. above converges to 2, the series in 1. and 4. are both divergent and it is not obvious yet but the series in 3. is divergent as well. Our next task is to give precise meanings to these terms for series. In order to do this we need some terminology. We know what it means for a sequence to converge, but we don't yet have a definition of convergence for series.

Convergence of a series

Definition 71

For a series $\sum_{n=1}^{\infty} a_n$, and for $k \geq 1$, let

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

Thus $s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3$ etc.
Then s_k is called the kth partial sum of the series, and the sequence $\{s_k\}_{k=1}^{\infty}$ is called the sequence of partial sums of the series.
If the sequence of partial sums converges to a limit s, the series is said to converge and s is called its sum. In this situation we can write $\sum_{n=1}^{\infty} a_n = s$. If the sequence of partial sums diverges, the series is said to diverge.

Convergence of a geometric series

Recall Example 2 above:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$$

In this example, for $k \geq 0$,

$$s_k = \sum_{n=0}^{k} \frac{1}{2^n} = 1 \quad + \quad \frac{1}{2} + \frac{1}{4} + \ldots \frac{1}{2^k}$$

$$\frac{1}{2}s_k = \sum_{n=0}^{k} \frac{1}{2^{n+1}} = \qquad \frac{1}{2} + \frac{1}{4} + \ldots \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

Then

$$s_k - \frac{1}{2}s_k = \frac{1}{2}s_k = 1 - \frac{1}{2^{k+1}} \Longrightarrow s_k = 2 - \frac{1}{2^k}.$$

So the sequence of partial sums has kth term $2 - \frac{1}{2^k}$. This sequence converges to 2 so the series converges to 2.

General geometric series

Consider the sequence of partial sums for the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

(This is a geometric series with initial term a and common ratio r.) The kth partial sum s_k is given by

$$\begin{array}{rclcccccc} s_k & = & \sum_{n=0}^{k} ar^n & = & a + & ar + & \dots & + & ar^k \\ rs_k & = & \sum_{n=0}^{k} ar^{n+1} & = & & ar + & ar^2 + & \dots + & ar^k & + & ar^{k+1} \end{array}$$

Then $(1-r)s_k = a - ar^{k+1} \Longrightarrow s_k = \dfrac{a(1-r^{k+1})}{1-r}$. If $|r| < 1$, then $r^{k+1} \to 0$ as $k \to \infty$, and the sequence of partial sums (hence the series) converges to $\dfrac{a}{1-r}$. If $|r| \geq 1$ the series is divergent.

The harmonic series is divergent

Theorem 72

The harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ *is divergent.*

Proof: Think of $\frac{1}{n}$ as the area of a rectangle of height $\frac{1}{n}$ and width 1, sitting on the interval $[n, n+1]$ on the x-axis. So the $\frac{1}{1}$ corresponds to a square of area 1 sitting on the interval $[1, 2]$, the term $\frac{1}{2}$ corresponds to a rectangle of area $\frac{1}{2}$ sitting on the interval $[2, 3]$ and so on.
The total area accounted for by these triangles is the sum of the harmonic series, and this exceeds the area accounted for by the improper integral

$$\int_1^{\infty}\frac{1}{x}\,dx.$$

From Section 1.5 we know that this area is infinite.

A necessary condition for convergence

Note: A necessary condition for the series

$$\sum_{n=1}^{\infty} a_n$$

to converge is that the sequence $(a_n)_{n=1}^{\infty}$ converges to 0; i.e. that $a_n \to 0$ as $n \to \infty$. If this does *not* happen, then the sequence of partial sums has no possibility of converging.

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The example of the harmonic series shows that the condition $a_n \to 0$ as $n \to \infty$ is not sufficient to guarantee that the series $\sum_{n=1}^{\infty}$ will converge.

Learning outcomes for Section 3.3

After studying this section you should be able to

- explain what an infinite series is and what it means for an infinite series to converge;
- Give examples of convergent and divergent series;
- show that the harmonic series is divergent;
- Use the "sigma" notation for sums.

Section 3.4: Introduction to power series

Definition 73

A power series in the variable x resembles a polynomial, except that it may contain infinitely many positive powers of x. It is an expression of the type

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots,$$

where each a_i is a number.

Example 74

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

is a power series.

Question: Can we think of a power series as a function of x?

Power Series as Functions

Define a "function" by

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

- If we try to evaluate this function at $x = 2$, we get a series of real numbers.

  $$f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \ldots$$

  This series is divergent, so our power series does not define a function that can be evaluated at 2.

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- If we try to evaluate this function at $x = 2$, we get a series of real numbers.

$$f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \dots$$

  This series is divergent, so our power series does not define a function that can be evaluated at 2.
- If we try evaluating at 0 (and allow that the first term x^0 of the power series is interpreted as 1 for *all* values of x), we get

$$f(0) = 1 + 0 + 0^2 + \dots = 1.$$

  So it does make sense to "evaluate" this function at $x = 0$.

- If we try evaluating at $x = \frac{1}{2}$, we get

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots$$

This is a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We know that if $|r| < 1$, such a series converges to the number $\dfrac{a}{1-r}$. In this case

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2,$$

and we have $f(\frac{1}{2}) = 2$.

So we can evaluate our function at $x = \frac{1}{2}$.

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$

A geometric series of this sort converges provided that the absolute value of its common ratio is less than 1. In general for any value of x whose absolute value is less than 1 (i.e. any x in the interval $(-1, 1)$), we find that $f(x)$ is a convergent geometric series, converging to $\dfrac{1}{1-x}$.

Conclusion: For values of x in the interval $(-1, 1)$ (i.e. $|x| < 1$), the function $f(x) = \frac{1}{1-x}$ coincides with the power series $\sum_{n=0}^{\infty} x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

The interval $(-1, 1)$ is called the interval of convergence of the power series, and 1 is the radius of convergence. We say that the power series representation of the function $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, for values of x in the interval $(-1, 1)$.

Which functions have power series representations?

Remark: The power series representation is not particularly useful if you want to calculate $\frac{1}{1-x}$ for some particular value of x, because this is easily done directly. However, if we could obtain a power series representation for a function like $\sin x$ and use it to evaluate (or approximate) $\sin(1)$ or $\sin(9)$ or $\sin(20)$, that might be of real practical use. These numbers are not easy to obtain directly because the definition of $\sin x$ doesn't tell us how to calculate $\sin x$ for a particular x - you can use a calculator of course but how does the calculator do it?

Questions: What functions can be represented by power series, and on what sorts of interval or subsets of $\mathbb{R}$? If a function could be represented by a power series, how would we calculate the coefficients in this series?

Maclaurin (or Taylor) series

Suppose that $f(x)$ is an infinitely differentiable function (this means that all the deriviatives of f are themselves differentiable), and suppose that f is represented by the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We can work out appropriate values for the coefficients c_n as follows.

- Put $x = 0$. Then $f(0) = c_0 + \sum_{n=1}^{\infty} c_n(0)^n \Longrightarrow f(0) = c_0$.
  The constant term in the power series is the value of f at 0.

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- Put $x = 0$. Then $f(0) = c_0 + \sum_{n=1}^{\infty} c_n (0)^n \implies f(0) = c_0$.
  The constant term in the power series is the value of f at 0.
- To calculate c_1, look at the value of the first derivative of f at 0, and differentiate the power series term by term. We expect

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

  Then we should have $f'(0) = c_1 + 2c_2 \times 0 + 3c_3 \times 0 + \cdots = c_1$. Thus

$$c_1 = f'(0).$$

- For c_2, look at the second derivative of f. We expect

$$f''(x) = 2(1)c_2 + 3(2)c_3 x + 4(3)c_4 x^2 + 5(4)c_5 x^3 + \ldots$$

Putting $x = 0$ gives $f''(0) = 2(1)c_2$ or

$$c_2 = \frac{f''(0)}{2(1)}.$$

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Putting $x = 0$ gives $f''(0) = 2(1)c_2$ or

$$c_2 = \frac{f''(0)}{2(1)}.$$

- For c_3, look at the third derivative $f^{(3)}(x)$. We have

$$f^{(3)}(x) = 3(2)(1)c_3 + 4(3)(2)c_4x + 5(4)(3)c_5x^2 + \ldots$$

Setting $x = 0$ gives $f^{(3)}(0) = 3(2)(1)c_3$ or

$$c_3 = \frac{f^{(3)}(0)}{3(2)(1)}$$

Coefficients of the Maclaurin Series

Continuing this process, we obtain the following general formula for c_n:

$$c_n = \frac{1}{n!} f^{(n)}(0).$$

Definition 75

For a positive integer n, the number n factorial, denoted $n!$ is defined by

$$n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1.$$

The number 0! (zero factorial) is defined to be 1.

Power series representation of $\sin x$

Write $f(x) = \sin x$, and write $\sum_{n=0}^{\infty} c_n x^n$ for the Maclaurin series of $\sin x$. Then

- $f(0) = \sin 0 = 0 \Longrightarrow c_0 = 0$

Power series representation of $\sin x$

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- $f'(0) = \cos 0 = 1 \Longrightarrow c_1 = 1$

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- $f''(0) = -\sin 0 = 0 \Longrightarrow c_2 = \frac{0}{2!} = 0$

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- $f^{(3)}(0) = -\cos 0 = -1 \Longrightarrow c_3 = \frac{-1}{3!} = -\frac{1}{6}$

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- $f''(0) = -\sin 0 = 0 \Longrightarrow c_2 = \frac{0}{2!} = 0$
- $f^{(3)}(0) = -\cos 0 = -1 \Longrightarrow c_3 = \frac{-1}{3!} = -\frac{1}{6}$
- $f^{(4)}(0) = \sin 0 = 0 \Longrightarrow c_4 = \frac{0}{4!} = 0$

Power series representation of $\sin x$

This pattern continues :

- If k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.

Thus the Maclaurin series for $\sin x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Note that this series only involves odd powers of x - this is not surprising because sin is an odd function; it satisfies $\sin(-x) = -\sin x$.

Power series representation of $\sin x$

This pattern continues :

- If k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.
- If k is odd and $k \equiv 1 \mod 4$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.

Thus the Maclaurin series for $\sin x$ is given by

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Power series representation of $\sin x$

This pattern continues :

- If k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.
- If k is odd and $k \equiv 1 \mod 4$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.
- If k is odd and $k \equiv 3 \mod 4$ then $f^{(k)}(0) = -\cos 0 = -1$ and $c_k = -\frac{1}{k!}$.

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Power series representations of $\sin x$ and $\cos x$

Theorem 76

For every real number x, the above series converges to $\sin x$.

Thus computing partial sums of this series gives us an effective way of approximating $\sin x$ for any real number x.

Exercise 77

Show that the Maclaurin series for $\cos x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k!)} x^{2k}.$$

(Note that this can be obtained by differentiating term-by-term the series for $\sin x$, as we would expect since $\frac{d}{dx}(\sin x) = \cos x$.)

Learning outcomes for Section 3.4

After studying this section you should be able to

- State the meaning of the term *power series*,
- Explain the concept of the *radius of convergence* of a power series,
- Calculate the coefficients in (an initial segment of) the Maclaurin series representation of a given function.