Section 3.2 : Sequences

Note: Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

Definition 55

A sequence is an infinite ordered list

$$a_1, a_2, a_3, \dots$$

- The items in list a_1, a_2 etc. are called terms (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence $a_1, a_2, \dots$ can be denoted by (a_n) or by $(a_n)_{n=1}^{\infty}$.
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty}$: $a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0,\ a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \dots$

$$0, 2, 0, 2, 0, 2, \dots$$

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2. $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \sin(\pi) = 0,\ a_3 = \sin(\frac{3\pi}{2}) = -1,\ a_4 = \sin(2\pi) = 0, \ldots.$

   $$1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

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3. $(\frac{1}{n}\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \frac{1}{n}\sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \frac{1}{2}\sin(\pi) = 0,\ a_3 = \frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3},\ a_4 = \frac{1}{4}\sin(2\pi) = 0, \dots$.

   $$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \dots$$

Visualising a sequence

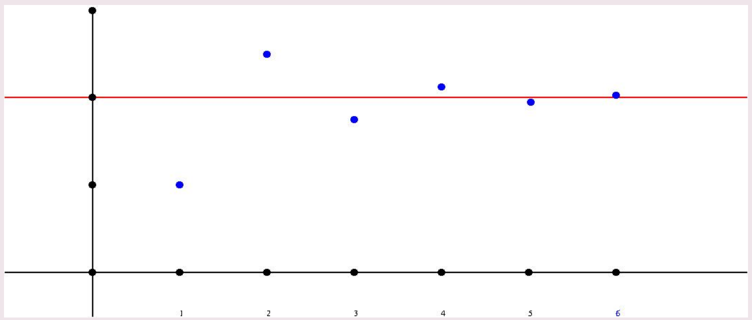
One way of visualizing a sequence is to consider is as a function whose domain is the set of natural numbers and think of its graph, which will be a collection of isolated points, one for each natural number.

Example 56

$\left(2 + \frac{(-1)^n}{2^{n-1}}\right)_{n=1}^{\infty}$. *Write* $a_n = 2 + \frac{(-1)^n}{2^{n-1}}$. *Then*

$$a_1 = 2 - \frac{1}{2^0} = 1,\ a_2 = 2 + \frac{1}{2} = \frac{5}{2},\ a_3 = 2 - \frac{1}{2^2} = \frac{7}{4}, a_4 = 2 + \frac{1}{2^3} = \frac{17}{8}.$$

Graphical representation of (a_n)*:*



The sequence $(2+(-1)^n\frac{1}{2^{n-1}})_{n=1}^{\infty}$

As n gets very large the positive number $\dfrac{1}{2^{n-1}}$ gets very small. By taking n as large as we like, we can make $\dfrac{1}{2^{n-1}}$ as small as we like.

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Hence for very large values of n, the number $2 + (-1)^n \frac{1}{2^{n-1}}$ is very close to 2. By taking n as large as we like, we can make this number as close to 2 as we like.

We say that the sequence converges to 2, or that 2 is the limit of the sequence, and write

$$\lim_{n\to\infty}\left(2 + (-1)^n \frac{1}{2^{n-1}}\right) = 2.$$

Note: Because $(-1)^n$ is alternately positive and negative as n runs through the natural numbers, the terms of this sequence are alternately greater than and less than 2.

Convergence of a sequence: "official" definitions

Definition 57

The sequence (a_n) converges to the number L (or has limit L) if for every positive real number ε (no matter how small) there exists a natural number N with the property that the term a_n of the sequence is within ε of L for all terms a_n beyond the Nth term. In more compact language:

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon \ \forall n > N.$$

Notes

- If a sequence has a limit we say that it converges or is convergent. If not we say that it diverges or is divergent.
- If a sequence converges to L, then no matter how small a radius around L we choose, there is a point in the sequence beyond which all terms are within that radius of L. So beyond this point, all terms of the sequence are *very close together* (and very close to L). Where that point is depends on how you interpret "very close together".

Ways for a sequence to be divergent

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 58

1. $(\max\{(-1)^n, 0\})_{n=1}^{\infty}$: 0, 1, 0, 1, 0, 1, ...
   This sequence alternates between 0 and 1 and does not approach any limit.

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2. *A sequence can be divergent by having terms that increase (or decrease) without limit.*
   $(2^n)_{n=1}^{\infty}$: 2, 4, 8, 16, 32, 64, ...

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3. *A sequence can have haphazard terms that follow no overall pattern, such as the sequence whose nth term is the nth digit after the decimal point in the decimal representation of* π.

Convergence is a precise concept!

Remark: The notion of a convergent sequence is sometimes described informally with words like "the terms get closer and closer to L as n gets larger". It is not true however that the terms in a sequence that converges to a limit L must get progressively closer to L as n increases.

Example 59

The sequence (a_n) is defined by

$$a_n = 0 \text{ if } n \text{ is even, } a_n = \frac{1}{n} \text{ if } n \text{ is odd.}$$

This sequence begins :

$$1,\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{5},\ 0,\ \frac{1}{7},\ 0,\ \frac{1}{9},\ 0, \dots$$

It converges to 0 although it is not true that every step takes us closer to zero.

Examples of convergent sequences

Example 60

Find $\lim_{n\to\infty} \dfrac{n}{2n-1}$.

Solution: As if calculating a limit as $x \to \infty$ of an expression involving a continuous variable x, divide above and below by n.

$$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n/n}{2n/n - 1/n} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}.$$

So the sequence $\left(\dfrac{n}{2n-1}\right)$ *converges to* $\frac{1}{2}$.

Bounded Sequences

As for subsets of $\mathbb{R}$, there is a concept of boundedness for sequences. Basically a sequence is bounded (or bounded above or bounded below) if the set of its terms, considered as a subset of $\mathbb{R}$, is bounded (or bounded above or bounded below).

Definition 61

The sequence (a_n) is bounded above if there exists a real number M for which $a_n \leq M$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded below if there exists a real number m for which $m \leq a_n$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded if it is bounded both above and below.

Example 62

The sequence (n) is bounded below (for example by 0) but not above.
The sequence $(\sin n)$ is bounded below (for example by -1) and above (for example by 1).

Convergent $\implies$ Bounded

Theorem 63

If a sequence is convergent it must be bounded.

Proof Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

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The set consisting of the first N terms of the sequence is a finite set : it has a maximum element M_1 and a minimum element m_1.

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Let $M = \max\{M_1, L + 1\}$ and let $m = \min\{m_1, L - 1\}$. Then (a_n) is bounded above by M and bounded below by m.

So our sequence is bounded.

Increasing and decreasing sequences

Definition 64

A sequence (a_n) is called increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called strictly increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$.
A sequence (a_n) is called strictly decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.

Definition 65

A sequence is called monotonic if it is either increasing or decreasing.
Similar terms: monotonic increasing, monotonic decreasing, monotonically increasing/decreasing.

Note: These definitions are not *entirely* standard. Some authors use the term *increasing* for what we have called *strictly increasing* and/or use the term *nondecreasing* for what we have called *increasing*.

Examples

1. An increasing sequence is bounded below but need not be bounded above. For example

$$(n)_{n=1}^{\infty} : 1,\ 2,\ 3,\ \dots$$

However, if a sequence is bounded and monotonic, it is convergent. This is the Monotone Convergence Theorem.

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2. A bounded sequence need not be monotonic. For example

$$((-1)^n) : -1,\ 1,\ -1,\ 1,\ -1,\ \ldots$$

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3. A convergent sequence need not be monotonic. For example

$$\left(\frac{(-1)^{n+1}}{n}\right)_{n=1}^{\infty} : 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \dots$$

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3. A convergent sequence need not be monotonic. For example

$$\left(\frac{(-1)^{n+1}}{n}\right)_{n=1}^{\infty} : 1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \dots$$

4. A monotonic sequence need not be convergent, as Example 1 above shows.

However, if a sequence is bounded and monotonic, it is convergent. This is the Monotone Convergence Theorem.

The Monotone Convergence Theorem

Theorem 66

If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.

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Choose a (very small) $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$, becasue L is the least upper bound.

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Choose a (very small) $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$, becasue L is the least upper bound.

This means there is some $N \in \mathbb{N}$ for which $L - \varepsilon < a_N$. Since L is an upper bound for $\{a_n : n \in \mathbb{N}\}$, this means

$$L - \varepsilon < a_N \leq L$$

Proof of the Monotone Convergence Theorem

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Since the sequence (a_n) is increasing and its terms are bounded above by L, every term after a_N is between a_N and L, and therefore between $L - \varepsilon$ and L. These terms are all within ε of L

Using the fact that our sequence is increasing and bounded, we have

- Identified L as the least upper bound for the set of terms in our sequence
- Showed that no matter how small an ε we take, there is a point in our sequence beyond which all terms are within ε of L.

This is exactly what it means for the sequence to converge to L.

An Example (from 2015 Summer Exam)

Example 67

A sequence (a_n) of real numbers is defined by

$$a_0 = 4, \ \ a_n = \frac{1}{2}(a_{n-1} - 2) \text{ for } n \geq 1.$$

1. *Write down the first four terms of the sequence.*
2. *Show that the sequence is bounded below.*
3. *Show that the sequence is montonically decreasing.*
4. *State why it can be deduced that the sequence is convergent, and determine its limit.*

Note: This is an example of a sequence that is defined recursively. This means that the first term is given and subsequent terms are defined (one by one) in terms of previous ones. We are not given a general formula for the nth term although one may exist.

Learning Outcomes for Section 3.2

After studying this section you should be able to

- Explain what a sequence is;
- State what it means for a sequence to be
  - convergent or divergent;
  - bounded or unbounded (above or below);
  - monotonic, increasing or decreasing.
- Give and/or identify examples of sequences with or without various properties (or combinations of properties) from the above list;
- State, prove and apply the Monotone Convergence Theorem;
- Analyze examples similar to Example 83.