MA203/MA283: Linear Algebra
Semester 2 2023-24
Practice Problem Sheet 4

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(v) = Av$, where A is the matrix $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -2 & 3 \\ -1 & 0 & -2 \end{bmatrix}$.

   What is the matrix of T with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$?

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v$.

   Show that there is no basis of $\mathbb{R}^2$ with respect to which the matrix of T is diagonal.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v$.

   Show that there is no basis of $\mathbb{R}^2$ with respect to which the matrix of T is diagonal.

4. Find a matrix P for which $P^{-1}AP = \text{diag}(1, -2, -10)$, where

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 0 & 4 \\ -4 & 2 & -10 \end{bmatrix}$$

   (Note that you do not need to calculate the characteristic polynomial of A to answer this!)

5. Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -2 \\ 0 & -1 & 2 \end{bmatrix}$, and hence find the eigenvalues of A. Find an eigenvector corresponding to each eigenvalue, and determine whether A is diagonalizable.

6. Give an example of a 2×2 matrix whose entries are all non-zero integers and whose eigenvalues are 2 and 4.

7. In $\mathbb{R}^2$, write $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as the sum of two vectors that are orthogonal to each other, one of which is parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

8. In $\mathbb{R}^2$, find the point on the line $y = 2x$ that is nearest to the point $(4, 5)$.

9. In $\mathbb{R}^3$, find $\text{proj}_v(u)$, where $v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $u = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

10. In $\mathbb{R}^3$, find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the plane $x + y + z = 1$.