MA203/MA283: LINEAR ALGEBRA
SEMESTER 2 2023-24
PRACTICE PROBLEM SHEET 3

1. Which of the following are subspaces of the vector space $M_2(\mathbb{R})$ consisting of all 2×2 matrices with real entries?
   (a) The set of all invertible matrices.
   (b) The set of all non-invertible matrices.
   (c) The set consisting only of the zero matrix.
   (d) The set $M_2(\mathbb{Q})$ of all matrices whose entries are rational.
   (e) The set of all matrices whose entry in the $(2,1)$-position is 0.
   (f) The set of all matrices whose four entries sum to 0.
   (g) The set of all matrices that have at least one entry equal to 0.
   (h) The set of all symmetric matrices (A is symmetric if $A = A^T$).

2. Which of the following are spanning sets of $M_2(\mathbb{R})$?
   (a) The set of all invertible matrices.
   (b) The set of all non-invertible matrices.
   (c) The set $M_2(\mathbb{Q})$ of all matrices whose entries are rational.
   (d) The set of all matrices whose entry in the $(2,1)$-position is 0.
   (e) The set of all matrices whose entry in the $(2,1)$-position is 1.
   (f) The set of all matrices whose four entries sum to 0.
   (g) The set of all matrices that have at least one entry equal to 0.

3. Determine whether or not each of the following is a spanning set of the set P_4 of all polynomials in $\mathbb{R}[x]$ of degree at most 3.
   (a) $\{x^3 + x^2, x^2 + x, x + 1\}$
   (b) $\{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1, 1\}$
   (c) $\{x^3 + x^2 + x - 1, x^3 + x^2 - x + 1, x^3 - x^2 + x + 1, -x^3 + x^2 + x + 1\}$
   (d) $\{x^3 + x^2 + x - 1, x^3 + x^2 - x + 1, x^3 - x^2 + x + 1, x^3 + x^2\}$

4. What is the dimension of the space of all symmetric matrices in $M_3(\mathbb{R})$?
   (Recall that A is symmetric if $A^T = A$).

5. Let $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$. What is the dimension of the space $v^\perp$ defined by

$$v^\perp = \{u \in \mathbb{R}^3 : u^T v = 0\}?$$

   Find a basis of $u^\perp$.

6. Determine whether each of the following subsets of $\mathbb{R}^3$ is linearly independent.
   (a) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \right\}$
   (b) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \right\}$

7. Let S be a linearly independent subset of a vector space V, and suppose that v is an element of V with $v \notin \langle S \rangle$. Show that $S \cup \{v\}$ is a linearly independent subset of V.

8. Extend the set
$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\}$$
to a basis of $\mathbb{R}^3$.

9. Show that $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

10. Find the change of basis matrix from the standard basis to the basis B above, and use it to find the B-coordinates of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (where the elements of B are ordered as in Question 9 above).