Lecture 19: Orthogonal Projection

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Orthogonality

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ (such as the ordinary scalar product).[1]

Definition We say that the vectors u and v are orthogonal (with respect to $\langle \cdot, \cdot \rangle$) if $\langle u, v \rangle = 0$.

These definitions are consistent with "typical" geometrically motivated concepts of distance and orthogonality.

Examples

1. $(2, 5)$ and $(5, -2)$ are orthogonal with respect to the ordinary scalar product in $\mathbb{R}^2$.
2. $\sin \pi x$ and $\cos \pi x$ are orthogonal with respect to the scalar product on the space of continuous functions on $[0, 1]$ defined in Lecture 18; this is saying that

$$\int_0^1 sin(\pi x)\,\cos(\pi x)\,dx = 0 \ \left(= \frac{1}{2\pi} \sin^2(\pi x) \Big|_0^1 \right).$$

[1] Recall that this means that $\langle u, v \rangle \in \mathbb{R}$ for all $u, v \in V$ (and the inner product $\langle \cdot, \cdot \rangle$ satisfies the symmetry, bilinearity, and non-negativity conditions from Lecture 18).

Orthogonal Projection

Lemma Let u and v be non-zero vectors in an inner product space V. Then it is possible to write (in a unique way) $v = au + v'$, where a is scalar and v' is orthogonal to u.

- If v is orthogonal to u, take $a = 0$ and $v' = v$.
- If v is a scalar multiple of u, take $au = v$ and $v' = 0$.
- Otherwise, to solve for a and v' in the equation $v = au + v'$ (with $u \perp v'$), take the inner product with u on both sides. Then

$$\langle u, v\rangle = a\langle u, u\rangle + 0 \Longrightarrow a = \frac{\langle u, v\rangle}{\langle u, u\rangle},\ v' = v - \frac{\langle u, v\rangle}{\langle u, u\rangle}u.$$

  We can verify directly that the two components in this expression are orthogonal to each other.

Example In $\mathbb{R}^2$, write $u = \binom{2}{1}$ and $v = \binom{6}{-2}$.

Orthogonal projection of one vector on another

Definition

For non-zero vectors u and v in an inner product space V, the vector $\frac{\langle u, v\rangle}{\langle u, u\rangle}u$ is called the projection of v on the 1-dimensional space spanned by u. It is denoted by $\mathrm{proj}_u(v)$ and it has the property that $v - \mathrm{proj}_u(v)$ is orthogonal to u.

Lemma

$\mathrm{proj}_u(v)$ *is the unique nearest element of the line $\langle u\rangle$ to v.*

Proof Let au be a scalar multiple of u. Then

$$d(au, v)^2 = \langle au - v, au - v\rangle = a^2\langle u, u\rangle - 2a\langle u, v\rangle + \langle v, v\rangle$$

Regarded as a quadratic function of a, this has a minimum when its derivative is 0, i.e. when $2a\langle u, u\rangle - 2\langle u, v\rangle = 0$, when $a = \frac{\langle u, v\rangle}{\langle u, u\rangle}$.

Orthogonal Bases (the Gram-Schmidt process)

Every finite-dimensional inner product space has an orthogonal basis[2]

To prove this, start with any basis $\{b_1, \dots, b_n\}$, and adjust the elements one by one (by subtracting off orthogonal projections of later vectors on earlier ones). The process ends with an orthogonal basis $\{v_1, \dots, v_n\}$.

1. Write $v_1 = b_1$, and $v_2 = b_2 - \text{proj}_{v_1}(b_2) = b_2 - \frac{\langle v_1, b_2 \rangle}{\langle v_1, v_1 \rangle} v_1$.

   Then the pairs b_1, b_2 and v_1, v_2 span the same space, and $v_1 \perp v_2$.
2. Write $v_3 = b_3 - \text{proj}_{v_1}(b_3) - \text{proj}_{v_2}(b_3)$.
   Then the sets $\{v_1, v_2, v_3\}$ and $\{b_1, b_2, b_3\}$ span the same space, and $v_3 \perp v_1$ and $v_3 \perp v_2$. To see this look at $\langle v_3, v_1 \rangle$ and $\langle v_3, v_2 \rangle$, noting that
   $$v_3 = b_3 - \frac{\langle b_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle b_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2.$$
3. Continue: at the kth step, form v_k by subtracting from b_k its projections on $v_1, \dots, v_n$. This leave v_k as the component of b_k that is orthogonal to all the previous v_i.

[2] This means a basis whose elements are all orthogonal to each other.

Orthogonal projection on a subspace

The result of this process is a basis $\{v_1, \ldots, v_n\}$ whose elements satisfy

$$\langle v_i, v_j \rangle = 0 \text{ for } i \neq j$$

We can adjust this basis to a orthonormal basis (consisting of orthogonal unit vectors) by replacing each v_i with its normalization $\hat{v}_i$.
From the Gram-Schmidt process, we have

Theorem

If V is a finite-dimensional inner product space, then V has an orthogonal (or orthonormal) basis.

Now let W be a subspace of V, and let $v \in V$. The orthogonal projection of v on W, denoted $\text{proj}_W(v)$, is defined to be the unique element u of W for which

$$v = u + v',$$

and $v' \perp w$ for all $w \in W$.