Lecture 13: Bases and Dimension

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[1] "Bases" is the plural of "basis"

Some definitions to recall

Let V be a vector space (e.g. $V = \mathbb{R}^n$). Let S be a (finite) subset of V.

1. S is a spanning set of V (or S spans V) if every element of V is a linear combination of the elements of S.
2. The span of S, denoted $\langle S \rangle$, is the set of all linear combinations of element of S, a subspace of V.
3. S is linearly independent if no element of S is a linear combination of the other elements of S.
   Equivalently, if no proper subset of S spans $\langle S \rangle$.
4. S is a basis of V if S is linearly independent AND S spans V.
   A basis is a minimal spanning set.
   A basis is a maximal linearly independent set.
5. Every finite spanning set of V contains a basis of V.
6. Every linearly independent subset of V can be extended to a basis of V (we have not proved this yet!).

The Steinitz Replacement Lemma

Theorem Let V be a vector space that has a basis with n elements. Then every linearly independent set with n elements is a basis of V.

Proof (for $n = 3$). Suppose $B = \{b_1, b_2, b_3\}$ is a basis of V, and let $\{y_1, y_2, y_3\}$ be a linearly independent subset of V.

1. $y_1 = a_1 b_1 + a_2 b_2 + a_3 b_3$ for scalars a_1, a_2, a_3, not all zero.
   We can assume (after maybe relabelling the b_i), that $a_1 \neq 0$.
   Then $\boxed{b_1 = a_1^{-1} y_1 - a_1^{-1} a_2 b_2 - a_1^{-1} a_3 b_3}$.
   So $b_1 \in \langle y_1, b_2, b_3 \rangle$ and $\{y_1, b_2, b_3\}$ spans V.
2. Now $y_2 \in \langle y_1, b_2, b_3 \rangle$ and y_2 is not a scalar multiple of y_1 (because $\{y_1, y_2, y_3\}$ is linearly independent).
   So b_2 (or b_3) has non-zero coefficient in any description of y_2 as a linear combination of y_1, b_2, b_3.
   Replace again: $\{y_1, y_2, b_2\}$ spans V.
3. Same reasoning: we can replace b_2 with y_3 to conclude $\{y_1, y_2, y_3\}$ spans V.

Conclusion $\{y_1, y_2, y_3\}$ is a basis of V.

Consequences of the replacement lemma

Theorem Let V be a vector space that has a basis with n elements. Then every linearly independent set with n elements is a basis of V.

1. If V has a spanning set with n elements, a linearly independent set in V cannot have more that n elements (by the same substitution argument).
2. If V has a linearly independent set with n elements, then a spanning set in V must have at least n elements (this is the same statement from the alternative viewpoint).

The number of elements in a linearly independent set cannot exceed the number in a spanning set. Every spanning set has at least as many elements as the biggest independent set.

Every basis has the same number of elements

Let V be a (finite dimensional) vector space, and let B and B' be bases of V. Then

- B is linearly independent and B' is a spanning set, so B has at most as many elements as B'.
- B is a spanning set and B' is linearly independent, so B has at least as many elements as $B'/$

It follows that B and B' have the same number of elements.

The dimension of V is the number of elements in a basis of V.

Exercise If $\dim V = n$, then every linearly independent subset of V has at most n elements, and every spanning set in V has at least n elements. Every spanning set with exactly n elements is a basis, and every linearly independent set with exactly n elements is a basis.

Note Every vector space that has a finite spanning set has a finite basis (since we can discard elements from a finite spanning set until a basis remains).

Examples

1. $\{1, x, x^2, x^3\}$ is a basis for the vector space P_3 of all polynomials of degree at most 3 with real coefficients. It is linearly independent because the only way to write the zero polynomial in the form $a_3x^3 + a_2x^2 + a_1x + a_0$ is by taking $a_0 = a_1 = a_2 = a_3 = 0$. Another basis of P_3, preferable for some applications, consists of the first four Legendre polynomials: $1, x, \frac{1}{2}(3x^2 - 1), \frac{1}{2}(5x^3 - 3x)$.

2. The row space of a $m \times n$ matrix is the subspace of $\mathbb{R}^n$ spanned by its rows. When we reduce the matrix to RREF, we are calculating a particular basis of its rowspace.

3. In $\mathbb{R}^2$, the reflection in the line $y = 2x$ sends $(1, 0)$ to $(-\frac{3}{5}, \frac{4}{5})$ and $(0, 1)$ to $(\frac{4}{5}, \frac{3}{5})$. Its (standard) matrix is $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$.
   The same reflection sends $(1, 2)$ to $(1, 2)$ and $(2, -1)$ to $(-2, 1)$. It is easier to describe it in terms of the basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.