Bounded and unbounded sets

Example

1. $\mathbb{Q}$ *is unbounded.*
2. $\mathbb{N}$ *is bounded below but not above.*
3. $(0,1)$, $[0,1]$, $[2,100]$ *are bounded.*
4. $\{\cos x : x \in \mathbb{R}\}$ *is bounded, since* $\cos x$ *can only have values between* -1 *and* 1.
5. *All finite subsets of* $\mathbb{R}$ *are bounded, and some infinite subsets are.*

Question: Is it possible for a bounded set to have the same cardinality as an unbounded set?

Open intervals

In our next example we show that the set of all the real numbers has the same cardinality as an open interval on the real line.
First we note that all open intervals have the same cardinality as each other.

Exercise 43

Show that the the open interval $(0, 1)$ *has the same cardinality as*

1. *The open interval* $(-1, 1)$
2. *The open interval* $(1, 2)$
3. *The open interval* $(2, 6)$.

The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example

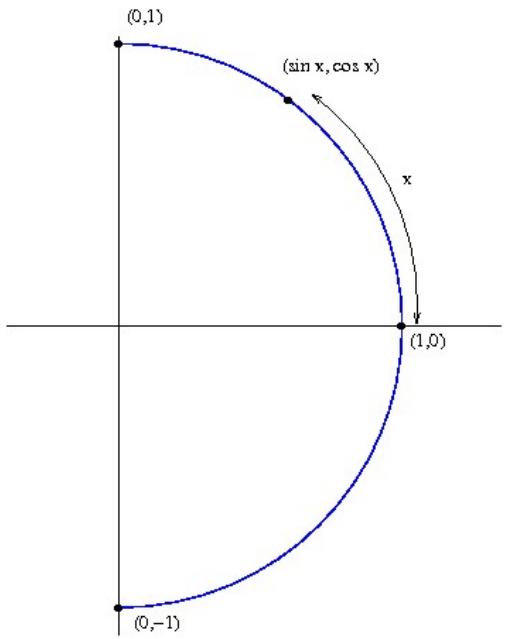
Show that $\mathbb{R}$ has the same cardinality as the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

In order to do this we have to establish a bijective correspondence between the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers.
An example of a function that provides us with such a bijective correspondence is familiar from calculus/trigonometry.

$(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$

Recall that for a number x in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\tan x$ is defined as follows: travel from $(1, 0)$ a distance $|x|$ along the circumference of the unit circle, anti-clockwise if x is positive and clockwise if x is negative. We arrive at a point which is in the right-hand side of the unit circle.

Now $\tan x$ is the slope of the line that connects the origin to this point (whose y and x-coordinates are $\sin x$ and $\cos x$ respectively).



$\tan x$ gives a bijection

Now $\tan 0 = 0$, and as x increases from 0 towards $\frac{\pi}{2}$, the line segment in question rotates about the origin into the first quadrant, its slope increases continuously from zero, without limit as x approaches $\frac{\pi}{2}$. So every positive real number is the tan of exactly one x in the range $(0, \frac{\pi}{2})$.

For the same reason, the values of $\tan x$ include every negative real number exactly once as x runs between 0 and $-\frac{\pi}{2}$.

$\tan x$ gives a bijection

Thus for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the correspondence

$$x \longleftrightarrow \tan x$$

establishes a bijection between the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers.

We conclude that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has the same cardinality as $\mathbb{R}$.

Note: This assertion is unrelated to the concept of countability discussed earlier.

Some Remarks

1. We don't know yet if $\mathbb{R}$ (or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) has the same cardinality as $\mathbb{N}$ - we don't know if $\mathbb{R}$ is countable.
2. The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ may seem like an odd choice for an example like this. However, note that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is in bijective correspondence with the interval $(-1, 1)$, via the function that just multiplies everything by $\frac{2}{\pi}$.

Learning outcomes for Section 2.3

This section contains some very challenging concepts. You will probably need to invest some serious intellectual effort in order to arrive at a good understanding of them. This is an effort worth making as it has the potential to really expand your view of what mathematics is about. After studying this section you should be able to

- Discuss the concept of bijective correspondence for infinite sets;
- Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality by exhibiting a bijective correspondence between them;
- Explain what is meant by a *countable* set and show that $\mathbb{Q}$ is countable;
- Exhibit a bijective correspondence between $\mathbb{R}$ and the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and hence show that $\mathbb{R}$ has the same cardinality as the interval (a, b) for any real numbers a and b with $a < b$.

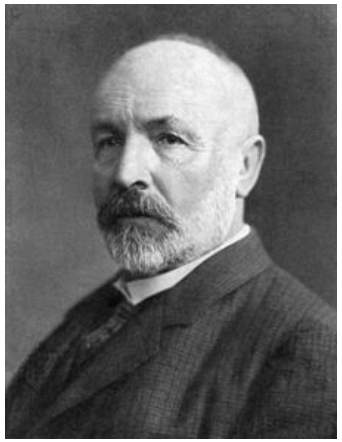
Section 2.4: $\mathbb{R}$ is uncountable

Our goal in this section is to show that the set $\mathbb{R}$ of real numbers is uncountable or non-denumerable; this means that its elements cannot be listed, or cannot be put in bijective correspondence with the natural numbers.

We saw at the end of Section 2.3 that $\mathbb{R}$ has the same cardinality as the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, or the interval $(-1, 1)$, or the interval $(0, 1)$. We will show that the open interval $(0, 1)$ is uncountable.

$(0, 1)$ is uncountable

This assertion and its proof date back to the 1890's and to Georg Cantor. The proof is often referred to as "Cantor's diagonal argument" and applies in more general contexts than we will see in these notes.



Georg Cantor : born in St Petersburg (1845), died in Halle (1918)

Theorem 44

The open interval $(0, 1)$ *is not a countable set.*

The open interval $(0, 1)$ is not a countable set

We recall precisely what this set is.

- It consists of all real numbers that are greater than zero and less than 1, or equivalently of all the points on the number line that are to the right of 0 and to the left of 1.
- It consists of all numbers whose decimal representation have only 0 before the decimal point (except $0.000\ldots$ which is equal to 0, and $0.99999\ldots$ which is equal to 1).
- Note that the digits after the decimal point may terminate in an infinite string of zeros, or may have a repeating pattern to their digits, or may not have either of these properties. The interval $(0, 1)$ includes all these possibilities.

A hypothetical bijective correspondence

Our goal is to show that the interval $(0, 1)$ cannot be put in bijective correspondence with the set $\mathbb{N}$ of natural numbers. Our strategy is to show that no attempt at constructing a bijective correspondence between these two sets can ever be complete; it can never involve *all* the real numbers in the interval $(0, 1)$ no matter how it is devised.

So imagine that we had a listing of the elements of the interval $(0, 1)$. Such a correspondence would have to look something like the following.

$\mathbb{N}$		$(0, 1)$
1	$\longleftrightarrow$	0.13567324 ...
2	$\longleftrightarrow$	0.10000000 ...
3	$\longleftrightarrow$	0.32323232 ...
4	$\longleftrightarrow$	0.56834662 ...
5	$\longleftrightarrow$	0.79993444 ...
$\vdots$		$\vdots$

Can this list be complete?

Our problem is to show that no matter how the right hand column is constructed, it can't contain every sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0,1)$
1	$\longleftrightarrow$	$0.\mathbf{1}3567324\ldots$
2	$\longleftrightarrow$	$0.1\mathbf{0}000000\ldots$
3	$\longleftrightarrow$	$0.32\mathbf{3}23232\ldots$
4	$\longleftrightarrow$	$0.568\mathbf{3}4662\ldots$
5	$\longleftrightarrow$	$0.7999\mathbf{3}444\ldots$
$\vdots$		$\vdots$

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$\mathbb{N}$		$(0,1)$
1	$\longleftrightarrow$	0.**1**3567324 ...
2	$\longleftrightarrow$	0.1**0**000000 ...
3	$\longleftrightarrow$	0.32**3**23232 ...
4	$\longleftrightarrow$	0.568**3**4662 ...
5	$\longleftrightarrow$	0.7999**3**444 ...
$\vdots$		$\vdots$

Look at the first digit after the decimal point in Item 1 in the list. If this is 1, write 2 as the first digit after the decimal point in x. Otherwise, write 1 as the first digit after the decimal point in x. So x differs in its first digit from Item 1 in the list.

Can this list be complete?

Our problem is to show that no matter how the right hand column is constructed, it can't contain every sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0,1)$
1	$\longleftrightarrow$	0.**1**3567324 ...
2	$\longleftrightarrow$	0.1**0**000000 ...
3	$\longleftrightarrow$	0.32**3**23232 ...
4	$\longleftrightarrow$	0.568**3**4662 ...
5	$\longleftrightarrow$	0.7999**3**444 ...
$\vdots$		$\vdots$

Look at the second digit after the decimal point in Item 2 in the list. If this is 1, write 2 as the second digit after the decimal point in x. Otherwise, write 1 as the second digit after the decimal point in x. So x differs in its second digit from Item 2 in the list.

Can this list be complete?

Our problem is to show that no matter how the right hand column is constructed, it can't contain every sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0,1)$
1	$\longleftrightarrow$	0.**1**3567324 ...
2	$\longleftrightarrow$	0.1**0**000000 ...
3	$\longleftrightarrow$	0.32**3**23232 ...
4	$\longleftrightarrow$	0.568**3**4662 ...
5	$\longleftrightarrow$	0.7999**3**444 ...
$\vdots$		$\vdots$

Look at the third digit after the decimal point in Item 3 in the list. If this is 1, write 2 as the third digit after the decimal point in x. Otherwise, write 1 as the third digit after the decimal point in x. So x differs in its third digit from Item 3 in the list.

Can this list be complete?

Our problem is to show that no matter how the right hand column is constructed, it can't contain every sequence of digits from 1 to 9. We do this by exhibiting an example of a sequence that can't possibly be there.

$\mathbb{N}$		$(0,1)$
1	$\longleftrightarrow$	0.**1**3567324 ...
2	$\longleftrightarrow$	0.1**0**000000 ...
3	$\longleftrightarrow$	0.32**3**23232 ...
4	$\longleftrightarrow$	0.568**3**4662 ...
5	$\longleftrightarrow$	0.7999**3**444 ...
$\vdots$		$\vdots$

Continue to construct x digit by digit in this manner. At the nth stage, look at the nth digit after the decimal point in Item n in the list. If this is 1, write 2 as the nth digit after the decimal point in x. Otherwise, write 1 as the nth digit after the decimal point in x. So x differs in its nth digit from Item n in the list.

Cantor's Diagonal Argument

What this process constructs is an element x of the interval $(0, 1)$ that does not appear in the proposed list. The number x is not Item 1 in the list, because it differs from Item 1 in its 1st digit, it is not Item 2 in the list because it differs from Item 2 in its 2nd digit, it is not Item n in the list because it differs from Item n in its nth digit.

Note:

- In our example, the number x would start $0.21111\ldots$.

Cantor's Diagonal Argument

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Note:

- In our example, the number x would start $0.21111\ldots$.
- According to our construction, our x will always have all its digits equal to 1 or 2. So not only have we shown that the interval $(0, 1)$ is uncountable, we have even shown that the set of all numbers in this interval whose digits are all either 1 or 2 is uncountable.

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- According to our construction, our x will always have all its digits equal to 1 or 2. So not only have we shown that the interval $(0, 1)$ is uncountable, we have even shown that the set of all numbers in this interval whose digits are all either 1 or 2 is uncountable.
- A challenging exercise : why would the same proof not succeed in showing that the set of rational numbers in the interval $(0, 1)$ is uncountable?

More on Cantor

Informally, Cantor's diagonal argument tells us that the "infinity" that is the cardinality of the real numbers is "bigger" than the "infinity" that is the cardinality of the natural numbers, or integers, or rational numbers. He was able to use the same argument to construct examples of infinite sets of different (and bigger and bigger) cardinalities. So he actually established the notion of infinities of different magnitudes.

More on Cantor

Informally, Cantor's diagonal argument tells us that the "infinity" that is the cardinality of the real numbers is "bigger" than the "infinity" that is the cardinality of the natural numbers, or integers, or rational numbers. He was able to use the same argument to construct examples of infinite sets of different (and bigger and bigger) cardinalities. So he actually established the notion of infinities of different magnitudes.

The work of Cantor was not an immediate hit within his own lifetime. It met some opposition from the finitist school which held that only mathematical objects that can be constructed in a finite number of steps from the natural numbers could be considered to exist. Foremost among the proponents of this viewpoint was Leopold Kronecker.

Kronecker

Leopold Kronecker (1823-1891)

God made the integers, all else is the work of man.

Kronecker

Leopold Kronecker (1823-1891)

God made the integers, all else is the work of man.

What good your beautiful proof on π? Why investigate such problems, given that irrational numbers do not even exist?

Hilbert

Cantor had influential admirers too, among them David Hilbert, who set the course of much of 20th Century mathematics in his address to the International Congress of Mathematicians in Paris in 1900.

David Hilbert (1862-1943)

No one shall expel us from the paradise that Cantor has created for us.

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David Hilbert (1862-1943)

No one shall expel us from the paradise that Cantor has created for us.

What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

The Continuum Hypothesis

Hilbert's address to the Paris Congress is one of the most famous mathematical lectures ever. In it he posed 23 unsolved problems, the first of which was Cantor's Continuum Hypothesis.

The Continuum Hypothesis proposes that every subset of $\mathbb{R}$ is either countable (i.e. has the same cardinality as $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Q}$) or has the same cardinality as $\mathbb{R}$.

This seems like a question to which the answer should be either a straightforward yes or no.

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It took the work of Kurt Gödel in the 1930s and Paul Cohen in the 1960s to reach the conclusion that the answer to this question of Cantor is undecidable. This means essentially that the standard axioms of set theory do not provide enough structure to determine the answer to the question.

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Both the Continuum Hypothesis and its negation are consistent with the working rules of mathematics. People who work in set theory can legitimately assume that either the Continuum Hypothesis is satisfied or not. Fortunately most of us can get on with our mathematical work without having to worry about this very often.

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References for this stuff:

1. Reuben Hersh, *What is Mathematics, Really?* Oxford University Press, 1997
2. Eugenia Cheng, *Beyond Infinity*, Profile Books, 2017

Note on Exam Questions

From the Summer 2015 exam:

Q2 (a) Give an example of

(i) An infinite subset of $\mathbb{R}$ in which every element is negative.

(ii) A subset of $\mathbb{R}$ that is bounded above but not below.

(iii) A subset of $\mathbb{R}$ that is infinite, countable and bounded.

Learning Outcomes for Section 2.4

After studying this section you should be able to

- Use Cantor's diagonal argument to prove that the interval $(0, 1)$ is uncountable.
- Make a few remarks about the history of this discovery.

Section 2.5 : The Completeness Axiom in $\mathbb{R}$

The rational numbers and real numbers are closely related.

- The set $\mathbb{Q}$ of rational numbers is countable and the set $\mathbb{R}$ of real numbers is not, and in this sense there are many more real numbers than rational numbers.
- However, $\mathbb{Q}$ is "dense" in $\mathbb{R}$. This means that every interval of the real number line, no matter how short, contains infinitely many rational numbers. This statement has a practical impact as well, which we use all the time.

Lemma 45

Every real number (whether rational or not) can be approximated by a rational number with a level of accuracy as high as we like.

Justification for this claim

- 3 is a rational approximation for π.
- 3.1 is a closer one.
- 3.14 is closer again.
- 3.14159 is closer still.
- 3.1415926535 is even closer than that,

and we can keep improving on this by truncating the decimal expansion of π at later and later stages.
If we want a rational approximation that differs from the true value of π by less than 10^{-20} we can truncate the decimal approximation of π at the 21st digit after the decimal point. This is what is meant by "a level of accuracy as high as we like" in the statement of the lemma.

Notes

1. The fact that all real numbers can be approximated with arbitrary closeness by rational numbers is used all the time in everyday life. Computers basically don't deal with all the real numbers or even with all the rational numbers, but with some specified level of precision. They really work with a subset of the rational numbers.
2. The sequence

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ 3.141592,\ \ldots$$

   is a list of numbers that are steadily approaching π. The terms in this sequence are increasing and they are approaching π. We say that this sequence converges to π and we will investigate the concept of convergent sequences in Chapter 3.
3. We haven't looked yet at the question of how the numbers in the above sequence can be calculated, i.e. how we can get our hands on better and better approximations to the value of the irrational number π. That's another thing that we will look at in Chapter 3.

Upper and Lower Bounds

The goal of this last section of Chapter 2 is to pinpoint one essential property of subsets of $\mathbb{R}$ that is not shared by subsets of $\mathbb{Z}$ or of $\mathbb{Q}$. We need a few definitions and some terminology in order to describe this.

Definition 46

Let S be a subset of $\mathbb{R}$. An element b of $\mathbb{R}$ is an upper bound for S if $x \leq b$ for all $x \in S$. An element a of $\mathbb{R}$ is a lower bound for S if $a \leq x$ for all $x \in S$.

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So an upper bound for S is a number that is to the right of all elements of S on the real line, and a lower bound for S is a number that is to the left of all points of S on the real line. Note that if b is an upper bound for S, then so is every number b' with $b < b'$. If a is a lower bound for S then so is every number a' with $a' < a$. So if S has an upper bound at all it has infinitely many upper bounds, and if S has a lower bound at all it has infinitely many lower bounds.

Upper and Lower Bounds

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Recall that

- S is bounded above if it has an upper bound,
- S is bounded below if it has a lower bound,
- S is bounded if it is bounded both above and below.

In this section we are mostly interested in sets that are bounded on at least one side.

Maximum and minimum elements

Definition 47

Let S be a subset of $\mathbb{R}$. If there is a number m that is both an element of S *and* an upper bound for S, then m is called the maximum element of S and denoted $\max(S)$.
If there is a number l that is both an element of S and a lower bound for S, then l is called the minimum element of S and denoted by $\min(S)$.

Notes

A set can have at most one maximum (or minimum) element.

Maximum and minimum elements

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If there is a number l that is both an element of S and a lower bound for S, then l is called the minimum element of S and denoted by $\min(S)$.

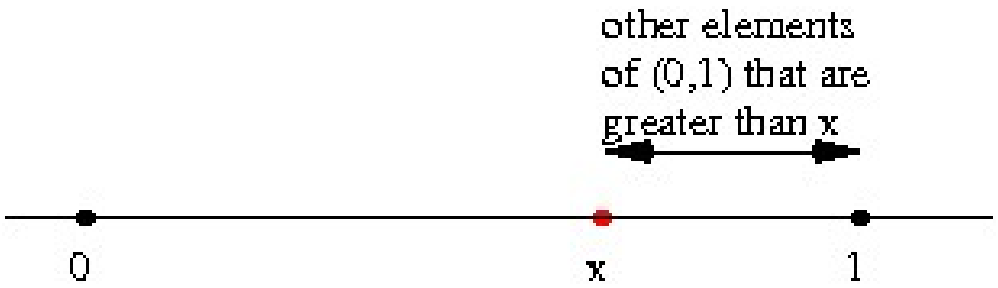
Notes

Pictorially, on the number line, the maximum element of S is the rightmost point that belongs to S, if such a point exists. The minimum element of S is the leftmost point on the number line that belongs to S, if such a point exists.

Not every set has a maximum element

There are basically two reasons why a subset S of $\mathbb{R}$ might fail to have a maximum element. First, S might not be bounded above - then it certainly won't have a maximum element.

Secondly, S might be bounded above, but might not contain an element that is an upper bound for itself. Take for example an open interval like $(0, 1)$. This set is certainly bounded above. However, take any element x of $(0, 1)$. Then x is a real number that is strictly greater than 0 and strictly less than 1. Between s and 1 there are more real numbers all of which belong to $(0, 1)$ and are greater than x. So x cannot be an upper bound for the interval $(0, 1)$.



Maximum and Minimum Elements

An open interval like $(0, 1)$, although it is bounded, has no maximum element and no minimum element.
An example of a subset of $\mathbb{R}$ that *does* have a maximum and a minimum element is a *closed* interval like $[2, 3]$. The minimum element of $[2, 3]$ is 2 and the maximum element is 3.

Remark : Every finite subset of $\mathbb{R}$ has a maximum element and a minimum element.