Partial fractions with long division

Example 31

Determine $\displaystyle\int \frac{x^3+3x+2}{x+1}\,dx$.

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing x^3+3x+2 by $x+1$.
We find that the quotient is x^2-x+4 and the remainder is -2. Hence

$$\frac{x^3+3x+2}{x+1} = x^2-x+4+\frac{-2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{x^3+3x+2}{x+1}\,dx &= \int x^2-x+4\,dx - 2\int \frac{1}{x+1}\,dx\\ &= \frac{1}{3}x^3-\frac{1}{2}x^2+4x-2\ln|x+1|+C.\end{aligned}$$

A Harder Example

Example 32

Determine $\displaystyle\int \frac{x+1}{(2x+1)^2(x-2)}\,dx.$

Solution: In this case the denominator has a repeated linear factor $2x+1$. It is necessary to include both $\dfrac{A}{2x+1}$ and $\dfrac{B}{(2x+1)^2}$ in the partial fraction expansion. We have

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-2}.$$

Then

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A(2x+1)(x-2) + B(x-2) + C(2x+1)^2}{(2x+1)^2(x-2)}.$$

and so

$$x+1 = A(2x+1)(x-2) + B(x-2) + C(2x+1)^2.$$

A Harder Example

$$x = 2: \quad 3 = C(5)^2 \qquad C = \frac{3}{25}$$

$$x = -\frac{1}{2}: \quad \frac{1}{2} = B\left(-\frac{5}{2}\right) \qquad B = -\frac{1}{5}$$

$$x = 0: \quad 1 = A(1)(-2) + B(-2) + C(1)^2 \qquad A = -\frac{6}{25}$$

Thus

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{-6/25}{2x+1} + \frac{-1/5}{(2x+1)^2} + \frac{3/25}{x-2}$$

and

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{6}{25}\int \frac{1}{2x+1}\,dx - \frac{1}{5}\int \frac{1}{(2x+1)^2}\,dx + \frac{3}{25}\int \frac{1}{x-2}\,dx.$$

A Harder Example

Call the three integrals on the right above $I_1,\ I_2,\ I_3$ respectively.

- $I_1 : \displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1).$

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

A Harder Example

Call the three integrals on the right above I_1, I_2, I_3 respectively.

- I_1 : $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1)$.
- I_2 : $\displaystyle\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}(+C_2)$.

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

A Harder Example

Call the three integrals on the right above I_1, I_2, I_3 respectively.

- I_1 : $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1)$.
- I_2 : $\displaystyle\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}(+C_2)$.
- I_3 : $\displaystyle\int \frac{1}{x-2}\,dx = \ln|x-2|(+C_3)$.

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

Learning outcomes for Section 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
  - direct methods;
  - suitably chosen substitutions;
  - integration by parts;
  - partial fraction expansions.

Section 1.5 : Improper Integrals

Suppose that $f(x)$ is a continuous function that satisfies

$$\lim_{x\to\infty} f(x) = 0;$$

for example $f(x) = e^{-x}$ has this property. Then we can consider the total area between the graph $y = f(x)$ and the X-axis, to the right of (for example) $x = 1$. This area is denoted by

$$\int_1^\infty f(x)\,dx$$

and referred to as an improper integral. For a given function, it is not clear whether the area involved is finite or infinite (if it is infinite, the improper integral is said to diverge or to be divergent). One question that arises is how we can determine if the relevant area is finite or infinite, another is how to calculate it if it is finite.

Definition of $\int_a^\infty f(x)\,dx$

Definition 33

If the function f is continuous on the interval $[a, \infty)$, then the *improper integral* $\displaystyle\int_a^\infty f(x)\,dx$ is defined by

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

provided this limit exists. In this case the improper integral is called convergent (otherwise it's divergent).

Similarly, if f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^a f(x)\,dx := \lim_{b\to-\infty} \int_b^a f(x)\,dx$$

Convergent or Divergent?

So to calculate an improper integral of the form $\int_1^\infty f(x)\,dx$ (for example), we first calculate

$$\int_1^b f(x)\,dx$$

for a general b. This will typically be an expression involving b. We then take the limit as $b \to \infty$.

Convergent or Divergent?

So to calculate an improper integral of the form $\int_1^\infty f(x)\,dx$ (for example), we first calculate

$$\int_1^b f(x)\,dx$$

for a general b. This will typically be an expression involving b. We then take the limit as $b \to \infty$.

Example 34

Show that the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *is divergent.*

Solution:

$$\int_1^b \frac{1}{x}\,dx = \ln x\big|_1^b = \ln b - \ln 1 = \ln b.$$

Since $\ln b \to \infty$ as $b \to \infty$, $\lim_{b\to\infty} \ln b$ does not exist and the integral diverges.

Another Example

Example 35

Evaluate $\int_{-\infty}^{-2} \frac{1}{x^2}\, dx$.

Solution:

$$\int_{b}^{-2} \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_{b}^{-2} = \frac{1}{2} + \frac{1}{b}$$

Then $\lim_{b\to-\infty} \left(\frac{1}{2} + \frac{1}{b}\right) = \frac{1}{2}$, and

$$\int_{-\infty}^{-2} \frac{1}{x^2}\, dx = \frac{1}{2}.$$

Yet another example

Example 36

Determine whether $\int_1^\infty e^{-2x}\,dx$ *is convergent or divergent, and evaluate it if it is convergent.*

Solution: $\int_1^b e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_1^b = -\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2}$. Then

$$\int_1^\infty e^{-2x}\,dx = \lim_{b\to\infty}(-\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2}) = \frac{e^{-2}}{2}.$$

So the integral is convergent and the enclosed area is $\frac{1}{2e^2}$.

Another type of improper integral

If the graph $y = f(x)$ has a vertical asymptote for a value of x in the interval $[c, d]$, these needs to be considered when computing the integral $\int_c^d f(x)\,dx$, since this integral describes the area of a region that is infinite in the vertical direction at the asymptote.

- If the vertical asymptote is at the left endpoint c, then we define

$$\int_c^d f(x)\,dx = \lim_{b \to c^+} \int_b^d f(x)\,dx.$$

Another type of improper integral

If the graph $y = f(x)$ has a vertical asymptote for a value of x in the interval $[c, d]$, these needs to be considered when computing the integral $\int_c^d f(x)\,dx$, since this integral describes the area of a region that is infinite in the vertical direction at the asymptote.

- If the vertical asymptote is at the right endpoint d, then we define

$$\int_c^d f(x)\,dx = \lim_{b \to d^-} \int_c^b f(x)\,dx.$$

Another type of improper integral

If the graph $y = f(x)$ has a vertical asymptote for a value of x in the interval $[c, d]$, these needs to be considered when computing the integral $\int_c^d f(x)\,dx$, since this integral describes the area of a region that is infinite in the vertical direction at the asymptote.

- If the vertical asymptote is at an interior point m of the interval $[c, d]$, then we define

$$\int_c^d f(x)\,dx = \int_c^m f(x)\,dx + \int_m^c f(x)\,dx,$$

  and the two improper integrals involving m are handled as above.

As in the case of improper integrals of the first type, these improper integrals are said to *converge* if the limits in question can be evaluated and to diverge if these limits do not exist. Divergence means that the area involved is infinite.

Final Example

Example 37

Determine whether the improper integral $\int_{-2}^{4} \frac{1}{x^2}\, dx$ *is convergent or divergent.*

Final Example

Example 37

Determine whether the improper integral $\displaystyle\int_{-2}^{4} \frac{1}{x^2}\,dx$ *is convergent or divergent.*

What makes this integral improper is the fact that the graph $y = \frac{1}{x^2}$ has a vertical asymptote at $x = 0$. Thus

$$\int_{-2}^{4} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{4} \frac{1}{x^2}\,dx$$

Final Example (continued)

For the first of these two integrals we have

$$\begin{aligned}
\int_{-2}^{0} \frac{1}{x^2}\,dx &= \lim_{b\to 0^-} \int_{-2}^{b} \frac{1}{x^2}\,dx \\
&= \lim_{b\to 0^-} \left(-\frac{1}{x}\right)\Big|_{-2}^{b} \\
&= \lim_{b\to 0^-} \left(-\frac{1}{b} + \frac{1}{2}\right)
\end{aligned}$$

Since $\lim_{b\to 0^-}(-\frac{1}{b})$ does not exist, the improper integral $\int_{-2}^{0} \frac{1}{x^2}\,dx$ diverges. This means that the area enclosed between the graph $y = \frac{1}{x^2}$ and the x-axis over the interval $[-2, 0]$ is infinite.

Final Example (continued)

For the first of these two integrals we have

$$\begin{aligned}\int_{-2}^{0} \frac{1}{x^2}\,dx &= \lim_{b\to 0^-} \int_{-2}^{b} \frac{1}{x^2}\,dx \\ &= \lim_{b\to 0^-} \left(-\frac{1}{x}\right)\Big|_{-2}^{b} \\ &= \lim_{b\to 0^-} \left(-\frac{1}{b}+\frac{1}{2}\right)\end{aligned}$$

Now that we know that the first of the two improper integrals in our problem diverges, we don't need to bother with the second. The improper integral $\int_{-2}^{4} \frac{1}{x^2}\,dx$ is divergent.

Note of Caution

In this last example, if we had not noticed the vertical asymptote at $x = 0$, we might have proceeded as follows:

$$\int_{-2}^{4} \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)\Big|_{-2}^{4} = -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}$$

This would be wrong! We should check that an integral is not improper before evaluating it in the manner above.

That's the end of Chapter 1

The first exam question (and first two homework assignments) are based on Chapter 1. The purpose of these items is to assess how well you have achieved the learning outcomes for Chapter 1. Section 1.6 of the lecture notes has some exam advice for Chapter 1, including sample "exam-type" questions with worked solutions and answers.

Things to do:

- Make sure you know how to use all the notation involving integrals, and distinguish definite, indefinite and improper integrals.
- Make sure you understand what the Fundamental Theorem of Calculus, so that you can state it and apply it to examples.
- Practise, practise, practise the techniques of integration (for example use a calculus textbook).

Section 2.1: The set $\mathbb{R}$ of real numbers

This section involves a consideration of properties of the set $\mathbb{R}$ of real numbers, the set $\mathbb{Q}$ of rational numbers, the set $\mathbb{Z}$ of integers and other related sets of numbers.
We will be interested in

- what is special about $\mathbb{R}$
- what distinguishes the real numbers from the rational numbers
- why the set of real numbers is such an interesting and important thing that there is a whole branch of mathematics (real analysis) devoted to its study.

What is $\mathbb{R}$?

There are at least two useful ways to think about what real numbers are. First we recall what *integers* are and what *rational numbers* are.

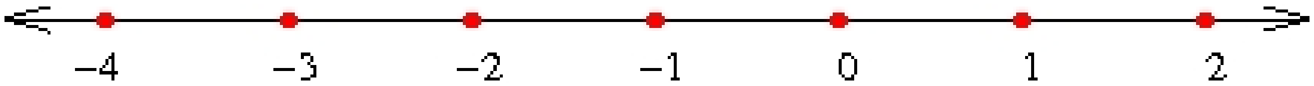
Integers are "whole numbers". The set of integers is denoted by $\mathbb{Z}$:

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

The notation "$\mathbb{Z}$" comes from the German word *Zahlen* (numbers).

The Integers $\mathbb{Z}$

On the number line, the integers appear as an infinite set of evenly spaced points. The integers are exactly those numbers whose decimal representations have all zeroes after the decimal point.



Note that the integers on the number line are separated by *gaps*. For example there are no integers in the chunk of the number line between $\frac{7}{5}$ and $\frac{63}{32}$.

Properties of the order on $\mathbb{Z}$

The set of integers is well-ordered.

Given any integer, it makes sense to talk about *the next* integer after that one.

For example, the next integer after 3 is 4.

To see why this property is something worth bothering about, it might be helpful to observe that the same property does not hold for the set $\mathbb{Q}$ of rational numbers described below.

The set $\mathbb{Q}$ of rational numbers

A rational number is a number that can be expressed as a fraction with an integer as the numerator and a non-zero integer as the denominator.

The set of all rational numbers is denoted by $\mathbb{Q}$ ($\mathbb{Q}$ is for *quotient*).

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, \ b \neq 0 \right\}.$$

The rational numbers

So $\mathbb{Q}$ includes such numbers as $\frac{5}{7}$, $-\frac{8}{16}$, $\frac{3141}{22445}$ and so on.

It includes all the integers, since any integer n can be written in the form of a fraction as $\frac{n}{1}$.

The rational numbers are exactly those numbers whose decimal representations either terminate (i.e. all digits are 0 from some point onwards) or repeat.

(Challenge: Prove this statement)

Note: The statement that $\mathbb{Q}$ includes all the integers can be written very concisely as $\mathbb{Z} \subset \mathbb{Q}$.

The geometry of $\mathbb{Z}$ and $\mathbb{Q}$

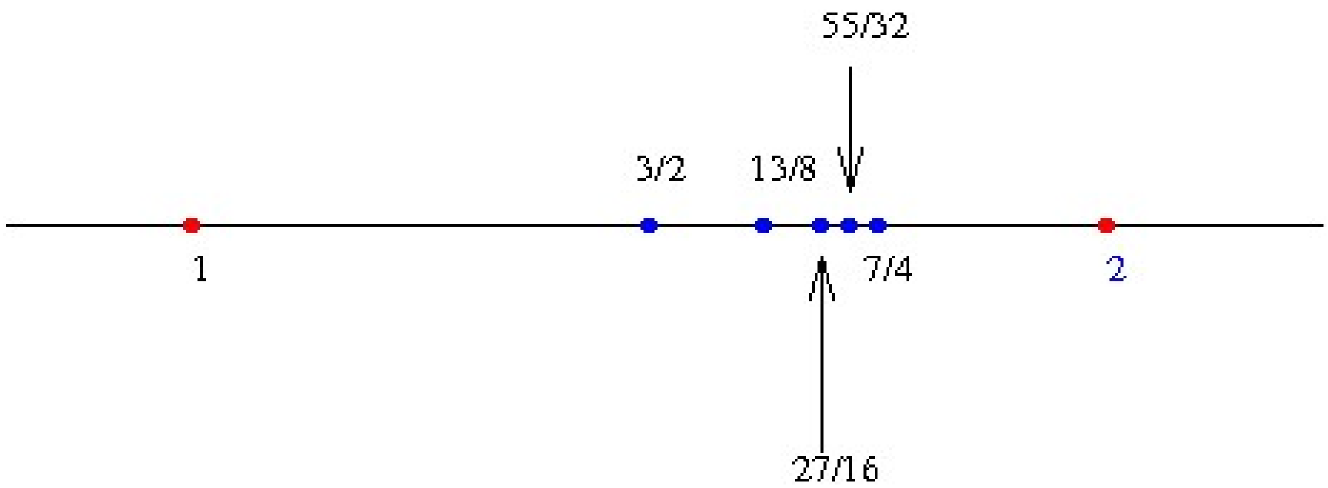
Since rational numbers can be written as quotients (or fractions) involving integers, the sets $\mathbb{Q}$ and $\mathbb{Z}$ are closely related.

However, on the number line these sets do not resemble each other at all. The integers are spaced out on the number line and there are gaps between them.

There are no stretches of the number line that are without rational numbers.

The rational numbers are densely packed

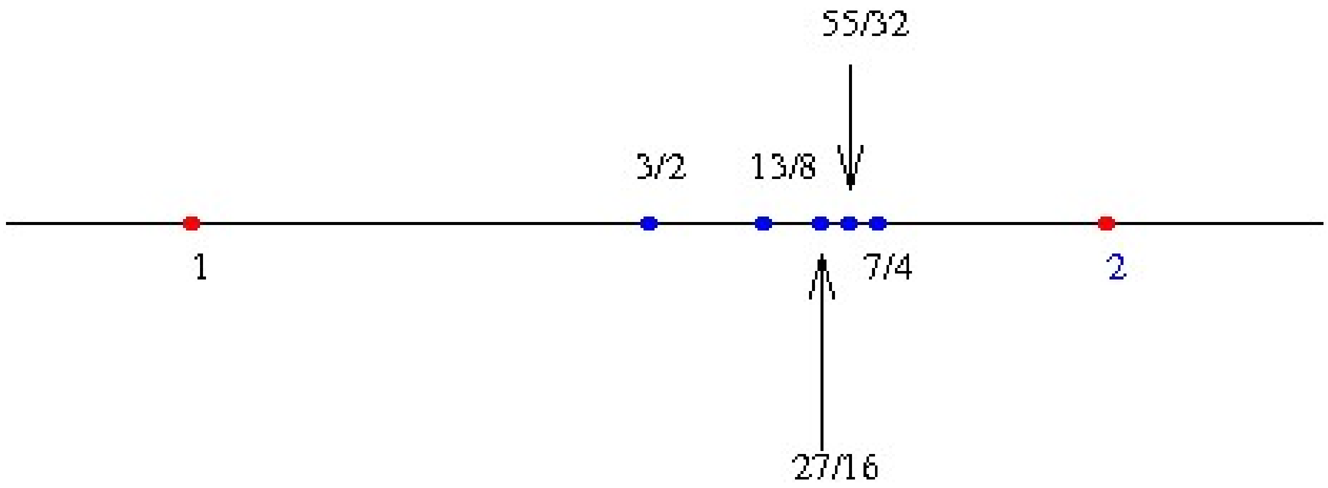
The diagram below is intended to show that the stretch of the number line between 1 and 2 contains infinitely many rational numbers.



The set of rational numbers is not well-ordered.
Given a particular rational number, there is no *next* rational number after it.

The rational numbers are densely packed

The diagram below is intended to show that the stretch of the number line between 1 and 2 contains infinitely many rational numbers.



Exercise 38

Choose a stretch of the number line - for example the stretch from $-\frac{7}{4}$ *to* $-\frac{11}{8}$ *(but pick your own example). Persuade yourself that your stretch contains infinitely many rational numbers.*