Lecture 7: More Matrix Algebra: Matrix Multplciation

February 1, 2024

Lecture 7: More Matrix Algebra: Matrix Multplication

Matrix Multiplication I (Matrix-vector multiplication)

We can sometimes multiply matrices by matrices.

Definition

Let A be a $m \times n$ matrix and let v be a column vector with n entries (a $n \times 1$ matrix). Then the matrix-vector product Av is the column vector obtained by taking the linear combination of the columns of A whose coefficients are the entries of v. It is a column vector with m entries.

Example

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix} = 7 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 41 \\ 33 \end{bmatrix}.$$

Another interpretation of Linear Systems

The linear system $\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$ may be interpreted as the matrix equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$$

A solution means an expression for $\begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix}$ is a linear combination of the three columns of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{bmatrix}$. The set of all such combinations is called the column space of A.

Matrix multplication II (Matrix-matrix multiplication)

Definition

Let A and B be matrices of size $m \times p$ and $p \times n$ respectively. Write $v_1, \dots, v_n$ for the columns of B. Then the product AB is the $m \times n$ matrices whose columns are $Av_1, \dots, Av_n$.

Matrix products are often described by their individual entries.
Suppose that A is a $m \times p$ matrix and B is a $p \times n$ matrix.
The entry in Row i and Column j of A is denoted A_{ij}.

The entry in the the (i, j)-position of AB (i.e. Row i and Column j) is the ith entry of the vector Av_j, where the vector v_j is Column j of B.

This is the linear combination of the ith entries of the columns of A (i.e. the entries of Row i of A, with coefficients from Column j of B). It is given by

$$(AB)_{ij} = A_{i1}B_{ij} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj} = \sum_{k=1}^{p} A_{ik}B_{kj}.$$

Matrix multiplication and the scalar product

The expression for $(AB)_{ij}$ above involves the scalar product of two vectors with p entries.

If A is $m \times p$ with rows $u_1, \dots, u_m$, and B is $p \times n$ with columns $v_1, \dots, v_n$, then the product AB is a table of values of scalar products of Rows of A with Columns of B.

$$AB = \begin{bmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 & \dots & u_1 \cdot v_n \\ u_2 \cdot v_1 & u_2 \cdot v_2 & \dots & u_2 \cdot v_n \\ \vdots & \vdots & & \vdots \\ u_m \cdot v_1 & u_m \cdot v_2 & \dots & u_m \cdot v_n \end{bmatrix}$$

Example $\begin{bmatrix} 1 & -2 & -3 \\ -5 & 2 & 0 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 6 & 2 \\ 0 & 3 \end{bmatrix} =$

$$\begin{bmatrix} 1(-4)+(-2)(6)+(-3)0 & 1(2)+(-2)(2)+(-3)3 \\ -5(-4)+2(6)+0(0) & -5(2)+2(2)+0(3) \end{bmatrix} = \begin{bmatrix} -16 & -11 \\ 32 & -6 \end{bmatrix}$$

Non-commutativity of matrix multiplication

- For matrices A and B, the products AB and BA are generally not equal, even if they are both defined and even if both have the same size.
- It can happen that one of AB and BA is defined and the other is not. For example if A is 3×4 and B is 4×2, then AB is a 3×2 matrix and BA is not defined.
- It can happen that AB and BA are both defined but have different sizes, for example of A is 3×4 and B is 4×3. Then AB is 3×3 and BA is 4×4.

Why are matrices so important?

Lots of reasons. Here is one.

> Because their algebraic operations encode interactions and relationships between quantities, that turn up in many contexts, not only within mathematics.

Our next example highlights how matrix multiplication can be hidden in very ordinary calculations that arise in everyday activities - where the matrices involved are just tables of numbers coming from some practical situation.

Exercise Make up another example like the one on the next slide, with a different context.

Making sense of matrix multplication

Example Let A be the 3×3 matrix formed by the table that gives the numbers of first year Humanities (H), Engineering (E) and Science (S) students in first year at Eigen University, in 2015, 2016 and 2017.

	H	E	S
2015	50	100	70
2016	60	80	80
2017	80	70	70

$$A = \begin{pmatrix} 50 & 100 & 70 \\ 60 & 80 & 80 \\ 80 & 70 & 70 \end{pmatrix}$$

Every first year student at Eigen University takes either Linear Algebra (LA) or Calculus (C) or both. The table below shows the numbers of ECTS credits completed annually in each, by students in each of the three subject areas.

	LA	C
H	10	0
E	15	15
S	20	10

$$B = \begin{pmatrix} 10 & 0 \\ 15 & 15 \\ 20 & 10 \end{pmatrix}$$

Now look at the meaning of the entries of the product AB.