Lecture 12: Linear Independence, Bases and Dimension

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Lecture 12: Linear Independence, Bases[1] and Dimension

[1] "Bases" is the plural of "basis"

Finite dimensional spaces

Definition A vector space V is finite dimensional if it contains a finite spanning set.

This means a set $\{v_1, \dots, v_k\}$ of elements, with the property that every element of V is a linear combination of $v_1, \dots, v_k$.

Examples

1. $\mathbb{R}^n$ is finite dimensional, with $\{e_1, \dots, e_n\}$ as a spanning set with n elements.
2. $M_{m\times n}(\mathbb{R})$ is finite dimensional, with $\{E_{ij}\}_{1\leq i\leq m,\ 1\leq j\leq n}$ as a spanning set with mn elements. The matrix E_{ij} has 1 in position (i, j) and zero in all other positions.
3. An example of vector space that is not finite dimensional is $\mathbb{R}[x]$, the space of all polynomials with coefficients in $\mathbb{R}$. If S is any finite set of polynomials, then the degree of a linear combination of elements of S can't exceed the highest degree of a polynomial in S.

Questions about Spanning Sets

1. Does $\mathbb{R}^3$ have a spanning set with fewer than three elements?
2. Does every spanning set of $\mathbb{R}^3$ have exactly three elements?
   NO (why not?)
3. Does every spanning set of $\mathbb{R}^3$ contain one with exactly three elements?
4. If V is a subspace of $\mathbb{R}^3$, does V have a spanning set with at most three elements?
5. If V is a proper subspace of $\mathbb{R}^3$ (i.e. not all of $\mathbb{R}^3$) does V have a spanning set with fewer than three elements?

Note A pair of vectors in $\mathbb{R}^3$ (if they are not scalar multiples of each other) span a plane. Adding a third vector (if it does not lie in this plane) gives a spanning set for all of $\mathbb{R}^3$.

Linear Dependence and Linear Independence

For a subset $\{v_1, \ldots, v_k\}$ of $\mathbb{R}^n$, suppose that v_k is a linear combination of $v_1, \ldots, v_{k-1}$. Then every linear combination of $v_1, \ldots, v_k$ is "already" a linear combination of $v_1, \ldots, v_{k-1}$ and

$$\langle v_1, \ldots, v_k \rangle = \langle v_1, \ldots, v_{k-1} \rangle.$$

If we are interested in the span of $\{v_1, \ldots, v_k\}$ we could throw away v_k and this would not change the span.

Definition A set of (at least two) vectors in $\mathbb{R}^n$ is linearly dependent if one of its elements is a linear combination of the others.
A set of vectors in R^n is linearly independent if it is not linearly dependent.[2]

Linear independence means that throwing away any element results in shrinking the span.

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[2] Small print: a set with just one vector is linearly independent, unless this vector is the zero vector. Any set that contains the zero vector is linearly dependent.

Example from Lecture 2

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array} \quad \begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three equations of the system, or the three rows of the original augmented matrix, formed a *linearly dependent set*. One row was eliminated by adding a linear combination of the other two. All the information in the system was contained in just (any) two of the three equations.

Meaning of linear independence A set is linearly independent if none of its elements is a linear combination of the others.

This definition makes conceptual sense, but to use it as a test for linear independence would mean checking it separately for every element of the set - not so efficient. We have an alternative formulation for this purpose, which is logically equivalent but maybe a bit harder to read.

Test for linear independence

To decide if the set $\{v_1, \dots, v_k\}$ is linearly independent, try to write the zero vector as a linear combination of the v_i:

$$\sum_{i=1}^{k} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0,$$

for scalars $a_1, \dots, a_k$. If $a_i = 0$ for every i is the only solution, then $v_1, \dots, v_k$ are linearly independent. If there is another solution, they are linearly dependent.

Example Decide whether $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^3$. Solution By row reduction we find

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies a = b = c = 0.$$

Conclusion The set is linearly independent.

What is a basis?

A **basis** of a vector space is a linearly independent spanning set.

- A basis is a minimal spanning set, one in which every element is needed, one that does not contain a smaller spanning set.
- Example: $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$.
  In general $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.
- $\{(1, 3), (1, 4)\}$ is a basis of $\mathbb{R}^2$.
- If S is a finite spanning set of a vector space V, then S contains a basis of V. If S is not linearly independent, then some $v \in S$ is a linear combination of the other elements of S. Throwing v away leaves a smaller set that still spans V. Repeat this step until a basis remains.

The Steinitz Replacement Lemma

Theorem Let V be a vector space that has a basis with n elements. Then every linearly independent set with n elements is a basis of V.

Proof (for $n = 3$). Suppose $B = \{b_1, b_2, b_3\}$ is a basis of V, and let $\{y_1, y_2, y_3\}$ be a linearly independent subset of V.

1. $y_1 = a_1 b_1 + a_2 b_2 + a_3 b_3$ for scalars a_1, a_2, a_3, not all zero.
   We can assume (after maybe relabelling the b_i), that $a_1 \neq 0$.
   Then $\boxed{b_1 = a_1^{-1} y_1 - a_1^{-1} a_2 b_2 - a_1^{-1} a_3 b_3}$.
   So $b_1 \in \langle y_1, b_2, b_3 \rangle$ and $\{y_1, b_2, b_3\}$ spans V.
2. Now $y_2 \in \langle y_1, b_2, b_3 \rangle$ and y_2 is not a scalar multiple of y_1 (because $\{y_1, y_2, y_3\}$ is linearly independent).
   So b_2 (or b_3) has non-zero coefficient in any description of y_2 as a linear combination of y_1, b_2, b_3.
   Replace again: $\{y_1, y_2, b_2\}$ spans V.
3. Same reasoning: we can replace b_2 with y_3 to conclude $\{y_1, y_2, y_3\}$ spans V.

Conclusion $\{y_1, y_2, y_3\}$ is a basis of V.