Lecture 11: Spanning sets and Linear Independence

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Subspaces

Definition A (non-empty) subset V of $\mathbb{R}^n$ is a subspace if

- It is closed under addition: $u + v \in V$ whenever $u \in V$ and $v \in V$.
- It is closed under scalar multiplication: $ku \in V$ whenever $u \in V$ and $k \in \mathbb{R}$.

Examples

1. $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$ is not a subspace of $\mathbb{R}^3$. The vectors $(1, 0, 0)$ and $(0, 1, 0)$ belong to this set but their sum $(1, 1, 0)$ does not.
2. $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) = 0\}$ is a subspace of $\mathbb{R}^3$.
3. $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (1, 2, 3) \neq 0\}$ is not a subspace of $\mathbb{R}^3$. For example, $(1, 4, 1)$ and $(-5, -2, -1)$ belong to this set but their sum $(-4, 2, 0)$ does not.
4. The kernel of any linear transformation is a subspace.
5. The image of any linear transformation is a subspace.

Exercise Prove these last two points.

How to make subspaces

Let $S = \{v_1, \dots, v_k\}$ be any (finite) subset of $\mathbb{R}^n$.

> The subset of $\mathbb{R}^n$ consisting of all linear combinations of the elements of S is a subspace of $\mathbb{R}^n$, denoted by $\langle S \rangle$ or $\langle v_1, v_2, \dots, v_k \rangle$ and called the linear span (or just span) of S.

Proof (that $\langle S \rangle$ is a subspace).
Closed under addition: let $u, v \in \langle S \rangle$. Then $u = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$, and $v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$, where the a_i and b_i are scalars. We need to show that $u + v \in \langle S \rangle$, which means showing that it is a linear combination of $v_1, \dots, v_k$. This is straightforward after everything has been set up, since $u + v = (a_1 + c_1)v_1 + (a_2 + c_2)v_2 + \cdots + (a_k + c_k)v_k$. So S is closed under addition.

Closed under scalar multiplication: let $u \in \langle S \rangle$ and $c \in \mathbb{R}$. We need to show that cu is a linear combination of $v_1, \dots, v_k$. We know that $u = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$, for scalars $a_1, \dots, a_k$. Then $cu = ca_1\, v_1 + ca_2\, v_2 + \cdots + ca_k\, v_k$, so $cu \in \langle S \rangle$.

Spanning Sets

Let V be a subspace of $\mathbb{R}^n$ (possibly V is all of $\mathbb{R}^n$). A subset S of V is called a spanning set of V if $\langle S \rangle = V$.
This means that every element of V is a linear combination of the elements of S.
Example The set $\{e_1, e_2, e_3\}$ is a spanning set of $\mathbb{R}^3$, where (as usual) $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This is saying that every element of $\mathbb{R}^3$ is a linear combination of e_1, e_2, e_3. For example

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2e_1 - 3e_2 + 4e_3.$$

Remark A set S of three column vectors in $\mathbb{R}^3$ is a spanning set of $\mathbb{R}^3$ if and only if each of e_1, e_2, e_3 is a linear combination of elements of S.
This occurs if and only if the 3×3 matrix whose columns are the vectors in S has an *inverse*.

Questions about Spanning Sets

1. Does $\mathbb{R}^3$ have a spanning set with fewer than three elements?
2. Does every spanning set of $\mathbb{R}^3$ have exactly three elements?
   NO (why not?)
3. Does every spanning set of $\mathbb{R}^3$ contain one with exactly three elements?
4. If V is a subspace of $\mathbb{R}^3$, does V have a spanning set with at most three elements?
5. If V is a proper subspace of $\mathbb{R}^3$ (i.e. not all of $\mathbb{R}^3$) does V have a spanning set with fewer than three elements?

Note A pair of vectors in $\mathbb{R}^3$ (if they are not scalar multiples of each other) span a plane. Adding a third vector (if it does not lie in this plane) gives a spanning set for all of $\mathbb{R}^3$.

Linear Dependence and Linear Independence

For a subset $\{v_1, \dots, v_k\}$ of $\mathbb{R}^n$, suppose that v_k is a linear combination of $v_1, \dots, v_{k-1}$. Then every linear combination of $v_1, \dots, v_k$ is "already" a linear combination of $v_1, \dots, v_{k-1}$ and

$$\langle v_1, \dots, v_k \rangle = \langle v_1, \dots, v_{k-1} \rangle.$$

If we are interested in the span of $\{v_1, \dots, v_k\}$ we could throw away v_k and this would not change the span.

Definition A set of (at least two) vectors in R^n is linearly dependent if one of its elements is a linear combination of the others.
A set of vectors in R^n is linearly independent if it is not linearly dependent.[1]

Linear independence means that throwing away any element results in shrinking the span.

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[1] Small print: a set with just one vector is linearly independent, unless this vector is the zero vector. Any set that contains the zero vector is linearly dependent.

More on Linear Independence

Example from Lecture 2

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array} \quad \begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three equations of the system, or the three rows of the original augmented matrix, formed a *linearly dependent set*. One row was eliminated by adding a linear combination of the other two. All the information in the system was contained in just (any) two of the three equations.

Meaning of linear independence A set is linearly independent if none of its elements is a linear combination of the others.

This definition makes conceptual sense, but to use it as a test for linear independence would mean checking it separately for every element of the set - not so efficient. We have an alternative formulation for this purpose, which is logically equivalent but maybe a bit harder to read.

Test for linear independence

To decide if the set $\{v_1, \dots, v_k\}$ is linearly independent, try to write the zero vector as a linear combination of the v_i:

$$\sum_{i=1}^{k} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0,$$

for scalars $a_1, \dots, a_k$. If $a_i = 0$ for every i is the only solution, then $v_1, \dots, v_k$ are linearly independent. If there is another solution, they are linearly dependent.

Example Decide whether $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent subset of $\mathbb{R}^3$. Solution By row reduction we find

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies a = b = c = 0.$$

Conclusion The set is linearly independent.