Example 22

Determine

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx.$$

Solution: Write

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx.$$

Now write $u=\sqrt{x}+1$. Then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} \Longrightarrow \frac{1}{\sqrt{x}}\,dx = 2du.$

Then

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx &= \int_{x=1}^{x=4} \frac{2}{u}\,du = \int_{u=2}^{u=3} \frac{2}{u}\,du = 2\ln u\Big|_2^3 \\ &= 2(\ln 3 - \ln 2) = 2\ln\frac{3}{2}.\end{aligned}$$

More Examples

Example 23

Determine $\int (1 - \cos t)^2 \sin t \, dt$

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Determine $\int (1 - \cos t)^2 \sin t \, dt$

Question: *How do we know what expression to extract and refer to as u?*
Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.

More Examples

Example 23

Determine $\int (1 - \cos t)^2 \sin t \, dt$

Question: *How do we know what expression to extract and refer to as u?*
Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.
There is no easy answer but with practice we can develop a sense of what might work. In this example the integrand involves the expression $1 - \cos t$ and also its derivative $\sin t$. This is what makes the substitution $u = 1 - \cos t$ effective for this problem.

NOTE: There are more examples of the substitution technique in the lecture notes.

1.4.2 : Integration by parts

In this section we discuss the technique of integration by parts, which is essentially a reversal of the product rule of differentiation.

Example 24

Find $\displaystyle\int x \cos x \, dx$.

Solution How could $x \cos x$ arise as a derivative?

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Example 24

Find $\int x \cos x \, dx$.

Solution How could $x \cos x$ arise as a derivative?
Well, $\cos x$ is the derivative of $\sin x$. So, if you were differentiating $x \sin x$, you would get $x \cos x$ but according to the product rule you would also get another term, namely $\sin x$.

Conclusion: $\int x \cos x \, dx = x \sin x + \cos x + C$.

Managing this process

What happened in this example was basically that the product rule was reversed. This process can be managed in general as follows. Recall from differential calculus that if u and v are expressions involving x, then

$$(uv)' = u'v + uv'.$$

Suppose we integrate both sides here with respect to x. We obtain

$$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx \Longrightarrow uv = \int u'v\,dx + \int uv'\,dx.$$

This can be rearranged to give the Integration by Parts Formula :

$$\int uv'\,dx = uv - \int u'v\,dx.$$

Here is the first example again, handled according to this scheme.

Example 25

Use the integration by parts technique to determine $\int x \cos x \, dx$.

Solution: Write

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array}$$

Then

$$\begin{aligned} \int x \cos x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \sin x - \int 1 \sin x \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

An antiderivative for $\ln x$

Example 26

Determine $\displaystyle\int \ln x \, dx$.

Solution: Let $u = \ln x$, $v' = 1$.
Then $u' = \frac{1}{x}$, $v = x$.

$$\begin{aligned}\int \ln x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \ln x - \int \frac{1}{x} x \, dx \\ &= x \ln x - x + C.\end{aligned}$$

Note: This example shows that sometimes problems which are not obvious candidates for integration by parts can be attacked using this technique.

Two Rounds of Integration by Parts

Sometimes two applications of the integration by parts formula are needed, as in the following example.

Example 27

Evaluate $\int x^2 e^x \, dx$.

Solution: Let $u = x^2,\ v' = e^x$. Then $u' = 2x,\ v = e^x$.

$$
\begin{aligned}
\int x^2 e^x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\
&= x^2 e^x - \int 2xe^x \, dx \\
&= x^2 e^x - 2\int xe^x \, dx.
\end{aligned}
$$

Let $I = \int xe^x \, dx$.

Let $I = \int xe^x \, dx$

To evaluate I apply the integration by parts formula a second time.

$$\begin{array}{ll} u = x & v' = e^x \\ u' = 1 & v = e^x. \end{array}$$

Then $I = \int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C$. Finally

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

An Example of Another Type

The next example shows another mechanism by which a second application of the integration by parts formula can succeed where the first is not enough.

Example 28

Determine $\int e^x \cos x \, dx$.

Solution Let

$$\begin{aligned} u &= e^x & v' &= \cos x \\ u' &= e^x & v &= \sin x. \end{aligned}$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

$\int e^x \cos x\, dx$ (continued)

For $\int e^x \sin x\, dx$: Let

$$\begin{array}{ll} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x. \end{array}$$

Then

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx,$$

and

$$\begin{aligned} \int e^x \cos x\, dx &= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\, dx \right) \\ \Longrightarrow 2 \int e^x \cos x\, dx &= e^x \sin x + e^x \cos x + C \\ \Longrightarrow \int e^x \cos x\, dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C \end{aligned}$$

A Definite Integral

Example 29

Evaluate $\int_0^1 (x+3)e^{2x}\,dx.$

Solution: Write $u = x+3,\ v' = e^{2x};\qquad u' = 1,\ v = \frac{1}{2}e^{2x}$

$$\begin{aligned}
\int_0^1 (x+3)e^{2x}\,dx &= \int uv'\,dx = (uv)\big|_0^1 - \int_0^1 u'v\,dx \\
&= \frac{x+3}{2}e^{2x}\bigg|_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx \\
&= \frac{x+3}{2}e^{2x}\bigg|_0^1 - \frac{1}{2}\times\frac{1}{2}e^{2x}\bigg|_0^1 \\
&= \frac{4}{2}e^2 - \frac{3}{2}e^0 - \frac{1}{4}e^2 + \frac{1}{4}e^0 = \frac{7}{4}e^2 - \frac{5}{4}.
\end{aligned}$$

We know how to integrate polynomial functions; for example

$$\int 2x^2 + 3x - 4\,dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$$

We also know that

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

and that

$$\int \frac{1}{x^n}\,dx = -\frac{1}{n-1}\frac{1}{x^{n-1}} + C,$$

for $n \neq 1$.

This section is about integrating rational functions; i.e. quotients in which the numerator and denominator are both polynomials.

Remark: If we were presented with the task of adding the expressions $\dfrac{2}{x+3}$ and $\dfrac{1}{x+4}$, we would take $(x+3)(x+4)$ as a common denominator and write

$$\begin{aligned}\frac{2}{x+3}+\frac{1}{x+4} &= \frac{2(x+4)}{(x+3)(x+4)}+\frac{1(x+3)}{(x+3)(x+4)}\\ &= \frac{2(x+4)+1(x+3)}{(x+3)(x+4)}=\frac{3x+11}{(x+3)(x+4)}.\end{aligned}$$

Question: Suppose we were presented with the expression $\dfrac{3x+11}{(x+3)(x+4)}$ and asked to rewrite it in the form $\dfrac{A}{x+3}+\dfrac{B}{x+4}$, for *numbers* A and B. How would we do it?

Another Question Why would we want to do such a thing?

Write

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}.$$

Then

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)} + \frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x+4A+3B}{(x+3)(x+4)}.$$

This means $3x+11 = (A+B)x + 4A + 3B$ for all x, which means

$$A + B = 3, \text{ and } 4A + 3B = 11.$$

Thus $B = 1$ and $A = 2$. So

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

An Alternative Method

We want

$$3x + 11 = A(x+4) + B(x+3),$$

for all real numbers x. If this statement is true for all x, then in particular it is true when $x = -4$. Setting $x = -4$ gives

$$-12 + 11 = A(0) + B(-1) \implies B = 1.$$

Setting $x = -3$ gives

$$-9 + 11 = A(1) + B(0) \implies A = 2.$$

Thus

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

Integration using partial fractions

Example 30

Determine $\displaystyle\int \frac{3x+11}{(x+3)(x+4)}\,dx.$

Solution : Write

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = \int \frac{2}{x+3}\,dx + \int \frac{1}{x+4}\,dx$$

Then

$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = 2\ln|x+3| + \ln|x+4| + C = \ln(x+3)^2 + \ln|x+4| + C.$$

Partial fractions with long division

Example 31

Determine $\displaystyle\int \frac{x^3+3x+2}{x+1}\,dx.$

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing x^3+3x+2 by $x+1$.
We find that the quotient is x^2-x+4 and the remainder is -2. Hence

$$\frac{x^3+3x+2}{x+1}=x^2-x+4+\frac{-2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{x^3+3x+2}{x+1}\,dx &= \int x^2-x+4\,dx-2\int\frac{1}{x+1}\,dx\\ &= \frac{1}{3}x^3-\frac{1}{2}x^2+4x-2\ln|x+1|+C.\end{aligned}$$

A Harder Example

Example 32

Determine $\displaystyle\int \frac{x+1}{(2x+1)^2(x-2)}\,dx.$

Solution: In this case the denominator has a repeated linear factor $2x+1$. It is necessary to include both $\dfrac{A}{2x+1}$ and $\dfrac{B}{(2x+1)^2}$ in the partial fraction expansion. We have

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-2}.$$

Then

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A(2x+1)(x-2)+B(x-2)+C(2x+1)^2}{(2x+1)^2(x-2)}.$$

and so

$$x+1 = A(2x+1)(x-2)+B(x-2)+C(2x+1)^2.$$

A Harder Example

$$x = 2: \quad 3 = C(5)^2 \qquad C = \frac{3}{25}$$

$$x = -\tfrac{1}{2}: \quad \tfrac{1}{2} = B\left(-\tfrac{5}{2}\right) \qquad B = -\frac{1}{5}$$

$$x = 0: \quad 1 = A(1)(-2) + B(-2) + C(1)^2 \qquad A = -\frac{6}{25}$$

Thus

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{-6/25}{2x+1} + \frac{-1/5}{(2x+1)^2} + \frac{3/25}{x-2}$$

and

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{6}{25}\int \frac{1}{2x+1}\,dx - \frac{1}{5}\int \frac{1}{(2x+1)^2}\,dx + \frac{3}{25}\int \frac{1}{x-2}\,dx.$$

A Harder Example

Call the three integrals on the right above $I_1, \ I_2, \ I_3$ respectively.

- I_1 : $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1).$

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

A Harder Example

Call the three integrals on the right above I_1, I_2, I_3 respectively.

- I_1 : $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1)$.
- I_2 : $\displaystyle\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}(+C_2)$.

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

A Harder Example

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- I_1 : $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|(+C_1)$.
- I_2 : $\displaystyle\int \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}(+C_2)$.
- I_3 : $\displaystyle\int \frac{1}{x-2}\,dx = \ln|x-2|(+C_3)$.

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

Learning outcomes for Section 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
  - direct methods;
  - suitably chosen substitutions;
  - integration by parts;
  - partial fraction expansions.