Section 1.3: The Fundamental Theorem of Calculus

In this section, we discuss the Fundamental Theorem of Calculus which establishes a crucial link between differential calculus and the problem of calculating definite integrals, or areas under curves.

At the end of this section, you should be able to explain this connection and demonstrate with some examples how the techniques of differential calculus can be used to calculate definite integrals.

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- If you know that at 10.00am yesterday the derivative of T was 0.5 ($^\circ$C/hr), then you know that the temperature was *increasing* by half a degree per hour at that time. However this does not tell you anything about what the temperature actually was at this time. If you know that by 10.00pm last night the derivative of the temperature was -2°C/hr you still don't know anything about what the temperature was at the time, but you know that it was cooling at a rate of 2 degrees per hour.

The "Area Accumulation" Function

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Let $s(t)$ be the distance travelled by the object after t seconds. From Section 1.1 we know that $s(t)$ is the area under the graph of $v(t)$ against t, between the vertical lines through 0 and t.
From the graph we can calculate $s(t) = 5t + t^2$.

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- $s(t) = 5t + t^2$ is the area under the graph $y = v(t)$, between 0 and t.
- Note $s'(t) = 5 + 2t = v(t)$.

Derivative of $s(t)$ is $v(t)$

Important Note: The function $s(t)$ associates to t the area under the graph $y = v(t)$ from time 0 to time t. As t increases (i.e. as time passes), this area increases (it represents the distance travelled which is obviously increasing). Note that the derivative of $s(t)$ is exactly $v(t)$.

$$s(t) = 5t + t^2; \; s'(t) = 5 + 2t = v(t).$$

We shouldn't really be surprised by this given the physical context of the problem: $s(t)$ is the total distance travelled at time t, and $s'(t)$ at time t is $v(t)$, the speed at time t. So this is saying that the instantaneous rate of change of the distance travelled at a particular moment is the *speed* at which the object is travelling at that moment - which makes sense.

The Fundamental Theorem of Calculus

Theorem 10

(The Fundamental Theorem of Calculus (FToC))
Let f be a (suitable) function, and let r be a fixed number. Define the area accumulation function A by

$$A(x) = \int_r^x f(t)\,dt.$$

This means: for a number x, $A(x)$ is the area enclosed by the graph of f and the x-axis, between the vertical lines through r and x.
The function A depends on the variable x, via the right limit in the definite integral. The Fundamental Theorem of Calculus tells us that the function f is exactly the derivative of this area accumulation function A. Thus

$$A'(x) = f(x).$$

An Example (Summer 2016)

Example 11

Define a function A by

$$A(x) = \int_1^x (\cos t + \sin t)^3 \, dt,$$

for $x \geq 1$*.*

1. *What is* $A(1)$*?*
2. *Show that the function A is decreasing at* $x = \pi$*.*
3. *Show that* $A'(\frac{3\pi}{4}) = 0$*.*

1. (A) $(\cos 1 + \sin 1)^3$ (B) 0 (C) $(\cos t + \sin t)^3$ (D) 1.
2. We need to show that the derivative of A is negative at $x = \pi$.
3. This is a direct application of the FTOC.

Notes on the Fundamental Theorem

1. We won't formally prove the FToC, but to get a feeling for what it says, think about how $A(x)$ changes when x moves a little to the right. What if $f(x) = 0$? What if $f(x)$ is large/small/positive/negative?

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2. The FToC is interesting because it connects differential calculus to the problem of calculating definite integrals, or areas under curves.
3. The FToC is useful because we know a lot about differential calculus. We can calculate the derivative of just about anything that can be written in terms of elementary functions. So we have a lot of theory about differentiation that is now relevant to calculating definite integrals as well.

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4. The FToC can be traced back to work of *Isaac Barrow* and *Isaac Newton* in the mid 17th Century.

Calculating Definite Integrals

Finally we see how to use the FToC to calculate definite integrals.

Example 12

Calculate $\int_1^3 t^3 - t^2\,dt.$

Solution: Imagine that r is some point to the left of 1, and that the function A is defined for $x \geq r$ by

$$A(x) = \int_r^x t^3 - t^2\,dt.$$

Then

$$\int_1^3 t^3 - t^2\,dt = A(3) - A(1);$$

This is the area under the graph that is to the left of 3 but to the right of 1.

Example of a definite integral calculation (continued)

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What we know about the function $A(x)$, from the Fundamental Theorem of Calculus, is that its derivative is given by $A'(x) = x^3 - x^2$. What function A has derivative $x^3 - x^2$?

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The derivative of x^4 is $4x^3$, so the derivative of $\frac{1}{4}x^4$ is x^3.

The derivative of x^3 is $3x^2$, so the derivative of $-\frac{1}{3}x^3$ is $-x^2$.

The derivative of $\frac{1}{4}x^4 - \frac{1}{3}x^3$ is $x^3 - x^2$.

Note : $\frac{1}{4}x^4 - \frac{1}{3}x^3$ is not the only expression whose derivative is $x^3 - x^2$.

For example $\frac{1}{4}x^4 - \frac{1}{3}x^3$ is another one, or anything of the form $\frac{1}{4}x^4 - \frac{1}{3}x^3 + C$, for any constant C. We only need one though.

Calculation of a definite integral

So: take $A(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3$. Then

$$\begin{aligned}
\int_1^3 t^3 - t^2\,dt &= A(3) - A(1) \\
&= \left(\frac{1}{4}(3^4) - \frac{1}{3}(3^3)\right) - \left(\frac{1}{4}(1^4) - \frac{1}{3}(1^3)\right) \\
&= \frac{81-1}{4} - \frac{27-1}{3} \\
&= \frac{34}{3}.
\end{aligned}$$

Fundamental Theorem of Calculus, Part 2

This technique is described in general terms in the following version of the Fundamental Theorem of Calculus :

Theorem 13

(Fundamental Theorem of Calculus, Part 2)
Let f be a function. To calculate the definite integral

$$\int_a^b f(x)\,dx,$$

first find a function F whose derivative is f, i.e. for which $F'(x) = f(x)$. (This might be hard). Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Learning outcomes for Section 1.3

After studying this section, you should be able to

- Describe what is meant by an "area accumulation function".
- State the Fundamental Theorem of Calculus.
- Use the FToC to solve problems similar to Example 11 in these slides.
- Describe the general strategy for calculating a definite integral.
- Evaluate simple examples of definite integrals, like the one in Example 12 in these slides.

To calculate

$$\int_a^b f(x)dx$$

1. Find a function F for which $F'(x) = f(x)$, i.e. find a function F whose derivative is f.
2. Evaluate F at the limits of integration a and b; i.e. calcuate $F(a)$ and $F(b)$. This means replacing x separately with a and b in the formula that defines $F(x)$.
3. Calculate the number $F(b) - F(a)$. This is the definite integral $\displaystyle\int_a^b f(x)\,dx$.

Of the three steps above, the first one is the hard one.

Notation

Recall the following notation : if F is a function that satisfies $F'(x) = f(x)$, then

$$F(x)\big|_a^b \text{ or } F(x)\big|_{x=a}^{x=b} \text{ means } F(b) - F(a).$$

Definition 14

Let f be a function. Another function F is called an antiderivative of f if the derivative of F is f, i.e. if $F'(x) = f(x)$, for all (relevant) values of the variable x.

So for example x^2 is an antiderivative of $2x$. Note that $x^2 + 1$, $x^2 + 5$ and $x^2 - 20e$ are also antiderivatives of $2x$. So we talk about an antiderviative of a function or expression rather that the antiderivative.

The Indefinite Integral

Definition 15

Let f be a function. The indefinite integral of f, written

$$\int f(x)\,dx$$

is the "general antiderivative" of f. If $F(x)$ is a particular antiderivative of f, then we would write

$$\int f(x)\,dx = F(x) + C,$$

to indicate that the different antiderivatives of f look like $F(x) + C$, where C may be any constant. (In this context C is often referred to as a constant of integration).

Example 16

Determine $\int \cos 2x \, dx$.

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By the chain rule, the derivative of $\sin 2x$ is in fact $2\cos 2x$.
So $\sin 2x$ is pretty close but it gives us twice what we want - we should compensate for this by taking $\frac{1}{2}\sin 2x$; its derivative is

$$\frac{1}{2}(2\cos 2x) = \cos 2x.$$

Conclusion: $\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C$

Powers of x

Example 17

Determine $\int x^n \, dx$

Important Note: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an antiderivative of x^n we have to reverse this process. This means that the index increases by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

The Integral of $\frac{1}{x}$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x}\, dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x}\, dx = \ln |x| + C.$$

This latter formula applies for all $x \neq 0$.