MA180/MA186/MA190 Calculus

Semester II 2022-23

Dr Rachel Quinlan

January 4, 2024

Lecturer
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Lectures: Wednesdays 10.00 in Anderson Lecture Theatre, Thursdays 10.00 in AMB-1022 (Fottrell Theatre)
Exception in Week 1: Monday 1.00 IT250, Tuesday 10.00 Anderson

Syllabus

The syllabus for Calculus consists of three chapters.

1. Integral Calculus

Definite Integrals and the Fundamental Theorem of Calculus. Techniques of Integration.

Learning Outcomes: You will be able to explain the connection between differential and integral calculus using the Fundamental Theorem of Calculus. You will be able to evaluate definite and indefinite integrals reliably, using a variety of techniques - this takes some practice. You will be able to communicate your ideas in a precise and clear manner.

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2. The Real Numbers

Properties of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. Finite and infinite sets. Different kinds of infinities. The order relation on the real numbers. Suprema and Infima. The completeness property of the real numbers.

Learning Outcomes: You will be able to distinguish between finite, countably infinite and uncountable sets of real numbers, explain these distinctions and provide examples to support your explanations. You will be able to explain the meanings of the terms supremum and infimum, analyze boundedness properties of given sets and provide your own examples of sets with prescribed properties. You will be able to write text (in sentences) that explains your understanding of these concepts.

Syllabus

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3. Sequences and Convergence

Sequences of real numbers. Convergent and divergent sequences. Boundedness and monotonicity.

Learning Outcomes: You will be able to explain the concept of convergence and its importance in mathematics, and discuss and relate various properties of sequences of real numbers. You will be able to determine with proof whether a given sequence of real numbers is convergent, and you will be able to provide your own examples of sequences with certain specified properties. You will be able to explain your ideas in writing.

Assessment

End-of-Semester Examinations:
One two-hour exam in Calculus and Algebra in the Summer Exam session, three questions from Algebra and three from Calculus (one corresponding to each chapter of the calculus course).

Continuous Assessment: A total of six assignments in Semester 2 on the OKUSON system. More details to follow.

Resources

Online resources for this course will be maintained mainly at http://rkq.ie/teaching/ma180_calculus/, linked from the "Calculus Semester 2" sections of the MA180, MA186 and MA190 Blackboard pages. These will include lecture notes which constitute the "text" for the course. You are expected to study the lecture notes, which are more detailed than the slides used in lectures. For further reading, the textbook "Calculus" by James Stewart is recommended, especially for Chapters 1 and 3 of the syllabus.

A Word of Advice

At this advanced level, Mathematics is more about understanding and explaining concepts and their logical connections than about doing calculations or "working out answers". This means that it involves learning a formal and technical language, which takes some practice. Don't be discouraged if it takes you some time to get used to this - your performance so far means that you are capable of it.

Section 1.1 Areas under curves - some examples

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(A) 600m.

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(A) 600m.

(B) 600m, obviously.

(C) 600m. Is this a trick question?

(D) Seriously? I got up at the crack of dawn and travelled back to Galway for this?

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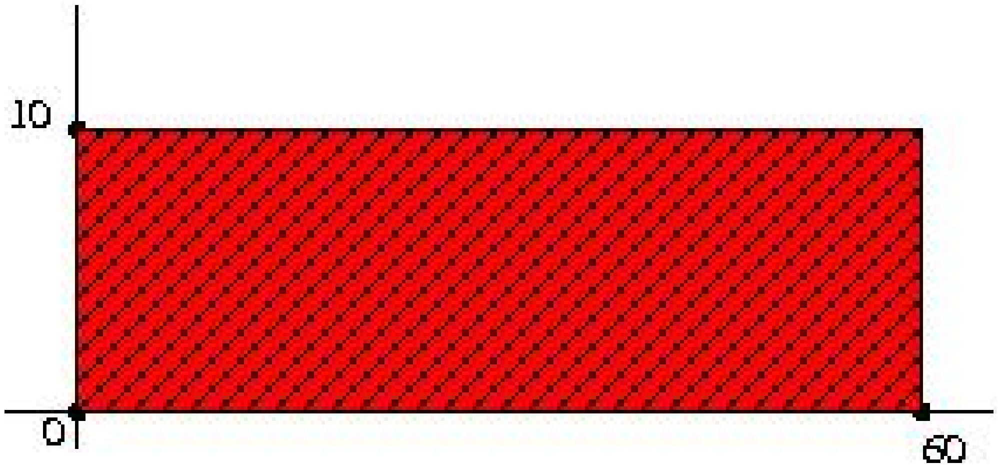
Graphical Interpretation

Suppose we draw a graph of the car's speed against time, where the x-axis is labelled in seconds and the y-axis in m/s. The graph is just the horizontal line $y = 10$ of course.



We label the time when we start observing the car's motion as $t = 0$ and the time when we stop as $t = 60$.
Note then that the total distance travelled (600m) is the area enclosed under the graph, between the x-axis, the horizontal line $y = 10$, and the vertical lines $x = 0$ (or time $t = 0$) and $x = 60$ marking the beginning and end of the period of observation.

Another Problem

Problem 2

This time the car travels at 10 m/s for the first 20 seconds, at 12 m/s for the next 20 seconds, and at 14 m/s for the last 20 seconds. What is the total distance travelled?

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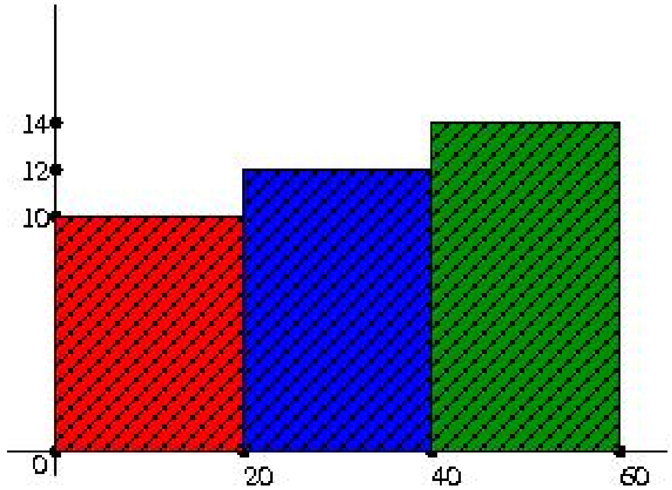


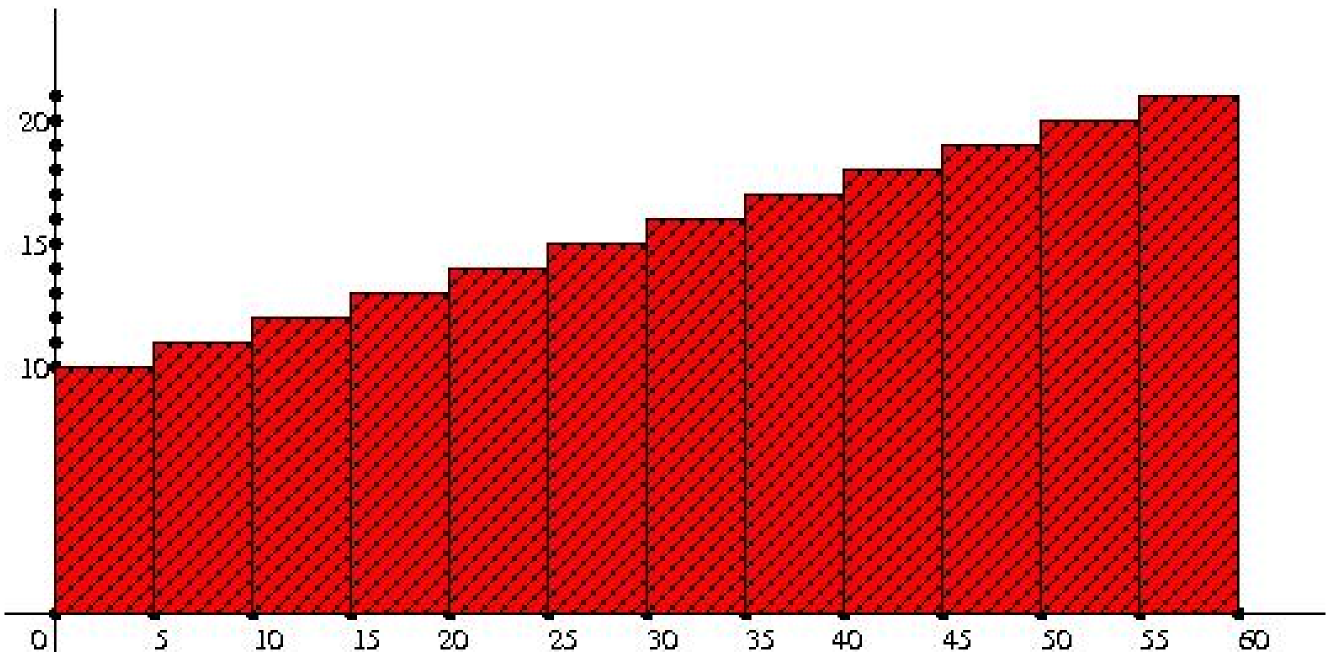
The total distance is again the area of the region enclosed between the lines $x = 0$, $x = 60$, the y-axis and the graph showing speed against time. The region whose area represents the distance travelled consists of three rectangles, all of width 20, and of heights 10, 12 and 14.

Another Problem

Problem 3

Same set up, but this time the car's speed is 10 m/s for the first 5 seconds, 11 m/s for the next 5, and so on, increasing by 1 m/s every five seconds so that the speed is 21 m/s for the last five seconds. Again the problem is to calculate the total distance travelled in metres.

The answer is left as an exercise, but this time the distance is the area indicated below.



A harder (but more realistic) problem

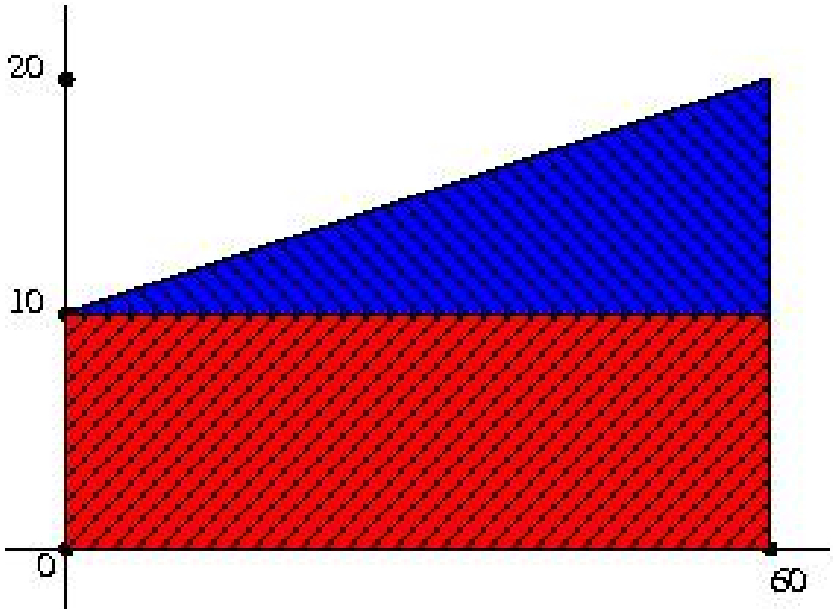
Problem 4

Again our car is travelling in one direction for one minute, but this time its speed increases at a constant rate from 10 m/s at the start of the minute, to 20 m/s at the end. What is the distance travelled?

Solution: This is a different problem, and more realistic. Because the speed is varying all the time this problem cannot be solved by just multiplying the speed by the time. However we can still consider the graph of the car's speed against time.

Distance travelled for continuously varying speed

Below is the graph of the speed against time.



If the total distance travelled is represented in this example, as in the case of constant speed, by the area under the speed graph between $t = 0$ and $t = 60$, then it is $60 \times 10 + \frac{1}{2}(60 \times 10) = 900$ metres.
Does this make sense?

Does this make sense?

Question 5

Just because the distance is given by the area under the graph when the speed is constant, how do we know the same applies in cases where the speed is varying continuously?

In the last problem, the speed increases steadily from 10 m/s to 20 m/s over the 60 seconds. We want to calculate the distance.

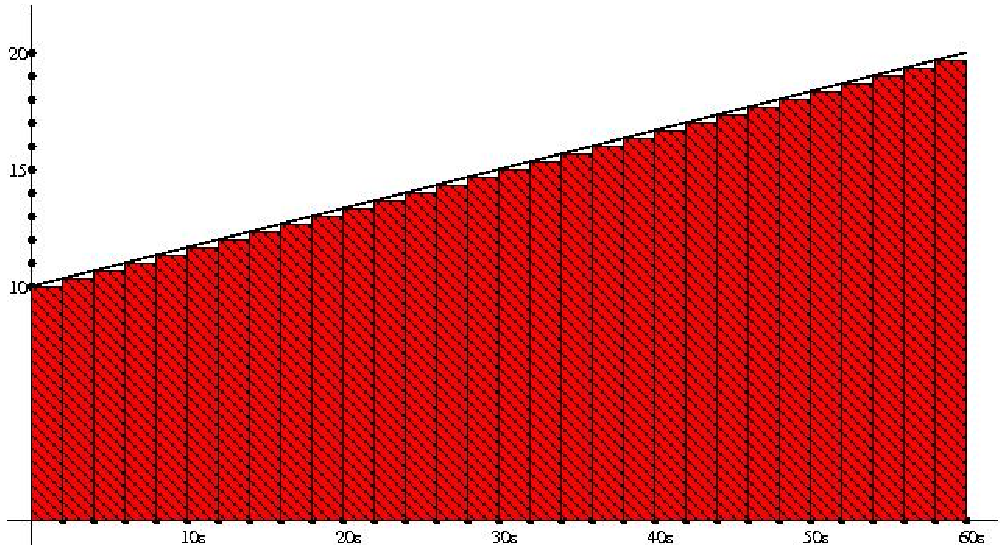
- Divide the one minute into 30 two-second intervals.
- At the start of the first two-second interval, the speed is 10 m/s. We make the simplifying assumption that the car travels at 10 m/s throughout these two seconds, covering 20 m in the first two seconds. This actually underestimates the true distance travelled in these two seconds, because in fact the speed is increasing from 20 m/s during this interval.

Continuously varying speed

- At the start of the 2nd two-second interval, the car has completed one-thirtieth of its acceleration from 10 m/s to 20 m/s, so its speed is
$$10 + \frac{10}{30} = 10\frac{1}{3}\text{m/s}.$$
Make the simplifying assumption that the speed remains constant at $10\frac{1}{3}$ m/s throughout the second two-second interval.
- Proceeding like this we would estimate that the car travels $890m$ during the one-minute period.

A lower Riemann Sum

The distance that we estimate using the assumption that the speed remains constant for each of the 30 two-second intervals is indicated by the area in red in the diagram below, where the black line is the true speed graph. Note that the red area includes all the area under the speed graph, except for 30 small triangles of base length 2 and height $\frac{1}{3}$.



Refining the Estimate

- Suppose now that we refine the estimate by dividing our minute of time into 60 one-second intervals and assuming the the speed remains constant for each of these, instead of into 30 two-second intervals. This would give us a total of $895m$ as the estimate for distance travelled (check this). What is the corresponding picture?
- Note that this still underestimates the distance travelled in each second, for the same reason as before. But this estimate is closer to the true answer than the last one, because this estimate takes into account speed increases every second, instead of every two seconds.
- If we used the same strategy but dividing our minute into 120 half-second intervals, we would expect to get a better estimate again.

The True Answer

As the number of subintervals increases and their width decreases, the red rectangles in the picture come closer and closer to filling all the area under the speed graph. The true distance travelled is the limit of these improving estimates, as the length of the subintervals approaches zero. This is exactly the area under the speed graph, between $x = 0$ and $x = 60$.

We can assert more confidently now that the answer is 900 metres.

Note for independent study: With some careful attention you can check that if you split the one minute into n subintervals each of length $\frac{60}{n}$ and estimate the distance travelled as above, your answer will be $900 - \frac{300}{n}$. The limit of this expression as $n \to \infty$ is 900.

What if the area is harder to calculate?

Problem 6

Again our car is travelling in one direction for one minute, but this time its speed v increases from 10 m/s to 20 m/s over the minute, according to the formula

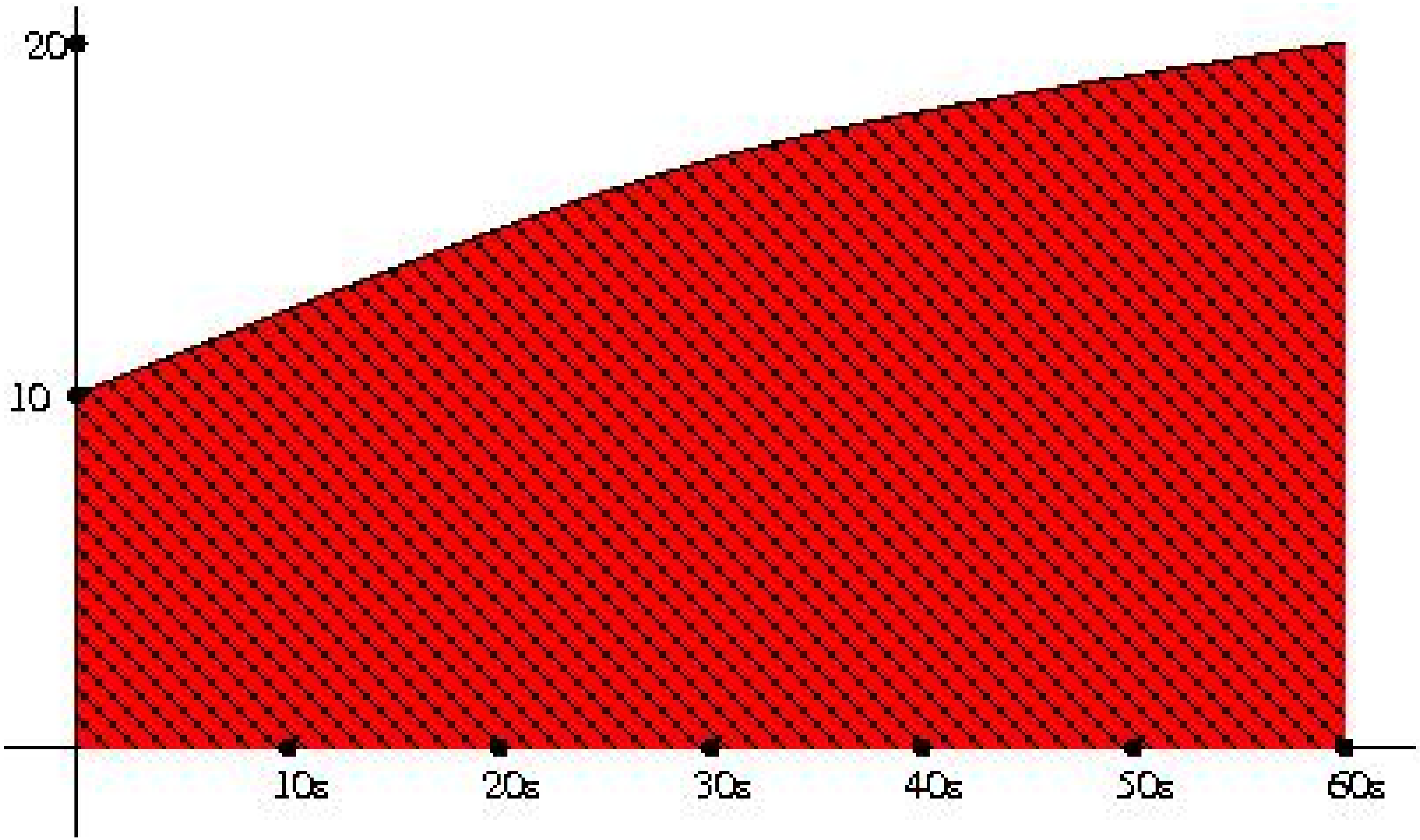
$$v(t) = 20 - \frac{1}{360}(60 - t)^2,$$

where t is measured in seconds, and $t = 0$ at the start of the minute. What is the distance travelled?

Note: The formula means that after t seconds have passed, the speed of the car in m/s is $20 - \frac{1}{360}(60 - t)^2$.

$v(t) = 20 - \frac{1}{360}(60 - t)^2$

Below is the graph of the speed (in m/s) against time (in s), with the area below it (between $t = 0$ and $t = 60$) coloured red.



A Harder Problem

The argument works in exactly the same way for this example, to persuade us that the distance travelled should be given by the area under the speed graph, between $t = 0$ and $t = 60$. This is the area that is coloured red in the picture.

Problem! The upper boundary of this area is a part of a parabola not a line segment. The region is not a combination of rectangles and triangles as in our earlier problems. We can't calculate its area using elementary techniques.

> So: what we need is a theory or a method that will allow us to calculate the area bounded by a section of the graph of a function and the x-axis, over a specified interval.

Same Idea: Other Examples

Important Note: The problem of calculating the distance travelled by an object from knowledge of how its speed is changing is just one example of a scientific problem that can be solved by calculating the area of a region enclosed between a graph and the x-axis. Here are just a few more examples.

1. The fuel consumption of an aircraft is a function of its speed. The total amount of fuel consumed on a journey can be calculated as the area under the graph showing speed against time.

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1. The fuel consumption of an aircraft is a function of its speed. The total amount of fuel consumed on a journey can be calculated as the area under the graph showing speed against time.
2. The energy stored by a solar panel is a function of the light intensity, which is itself a function of time. The total energy stored in one day can be modelled as the area under a graph of the light intensity against time for that day.

Same Idea : Other Examples

3. The volume of (for example) a square pyramid can be interpreted as the area of a graph of its horizontal cross-section area against height above the base.

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4. In medicine, if a drug is administered intravenously, the quantity of the drug that is in the person's bloodstream can be calculated as the area under the graph of a function that depends both on the rate at which the drug is administered and on the rate at which it breaks down.

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5. The concept of area under a graph is widely used in probability and statistics, where for example the probability that a randomly chosen person is aged between 20 and 30 years is the area under the graph of the appropriate probability density function, over the relevant interval.

Learning Outcomes from Section 1.1

After studying this section, you should be able to

- Explain, with reference to examples, why it is of interest to be able to calculate the area under the graph of a function over some specified interval.
- Explain how such an area can be approximated using rectangles, and how closer approximations can be obtained by taking narrower rectangles and using more of them.
- Solve simple problems similar to Problems 3 and 4 in this section.

Section 1.2: The Definite Integral

Goal of this section:
In the last section we concluded that we would like a theory for discussing (and hopefully calculating) areas enclosed between the graphs of known functions and the x-axis, within specified intervals.

Such a theory does exist and it forms a large part of what is called integral calculus.

In order to develop and use this theory we need a technical language and notation for talking about areas under curves.
The goal of this section is to understand this notation and be able to use it.

An Example

Example 7

Suppose that f is the function defined by $f(x) = x^2$.

Note that $f(x)$ is positive when $1 \leq x \leq 3$.

This means that in the region between the vertical lines $x = 1$ and $x = 3$, the graph $y = f(x)$ lies completely above the x-axis.

The area of the region enclosed between the graph $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 3$ is called the definite integral of x^2 from $x = 1$ to $x = 3$, and denoted by

$$\int_1^3 x^2\,dx.$$

Note that in general for a function f and numbers a and b, the definite integral $\int_a^b f(x)\,dx$ is a number.

Definition of the definite integral

Definition 8

Let a and b be fixed real numbers, with $a < b$. Let f be a function for which it makes sense to talk about the area enclosed between the graph of f and the x-axis, over the interval from a to b. Then the definite integral from a to b of f, denoted

$$\int_a^b f(x)\,dx$$

is defined to be the number obtained by subtracting the area enclosed below the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$ from the area enclosed above the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$.

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So the definite integral is essentially the area enclosed between the graph and the x-axis in the relevant interval, except that area below the x-axis is considered to be negative.

Notes

1. In our definition, what is meant by the phrase "for which it makes sense to talk about the area enclosed between the graph of f and the x-axis" is (more or less) that the graph $y = f(x)$ is not just a scattering of points, but consists of a curve or perhaps more than one curve. There is a formal theory about "integrable functions" that makes this notion precise.

2. Note on Notation

The notation surrounding definite integrals is a bit unusual. This note explains the various components involved in the expression

$$\int_a^b f(x)\,dx.$$

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Notes (continued)

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   - "$\int$" is the integral sign.
   - The "dx" indicates that f is a function of the variable x, and that we are talking about area between the graph of $f(x)$ against x and the x-axis.
   - The "$f(x)$" in $\int_a^b f(x)\,dx$ is called the integrand. It is the function whose graph is the upper (or lower) boundary of the region whose area is being described.

Notes (continued)

- The numbers a and b are respectively called the lower and upper (or left and right) limits of integration. They determine the left and right boundaries of the region whose area is being described.
  In the expression $\int_a^b f(x)\,dx$, the limits of integration a and b are taken to be values of the variable x - this is included in what is to be interpreted from "dx".
  If there is any danger of ambiguity about this, you can write

$$\int_{x=a}^{x=b} f(x)\,dx \quad \text{instead of} \quad \int_a^b f(x)\,dx.$$

  Please do not confuse this use of the word "limit" with its other uses in calculus.

Some Historical Remarks

The notation that is currently in use for the definite integral was introduced by Gottfried Leibniz around 1675. The rationale for it is as follows :

- Areas were estimated as we did in Section 1.1. The interval from a to b would be divided into narrow subintervals, each of width Δx.

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- The name x_i would be given to the left endpoint of the ith subinterval, and the height of the graph above the point x_i would be $f(x_i)$.
- So the area under the graph on this ith subinterval would be approximated by that of a rectangle of width Δx and height $f(x_i)$.

Some Historical Remarks (continued)

- The total area would be approximated by the sum of the areas of all of these narrow rectangles, which was written as

$$\sum f(x_i)\Delta x.$$

  The accuracy of this estimate improves as the width of the subintervals gets smaller and the number of them gets larger; the true area is the limit of this process as $\Delta x \to 0$.

For more information on the history of calculus and of mathematics generally, see `http://www-history.mcs.st-and.ac.uk/index.html`.

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- The notation "dx" was introduced as an expression to replace Δx in this limit, and the integral sign $\int$ is a "limit version" of the summation sign $\sum$. The integral symbol itself is based on the "long s" character which was in use in English typography until about 1800.

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Learning Outcome for Section 1.2

Just to be able to use the notation for definite integrals correctly.
This notation is admittedly a bit obscure but with careful attention you can certainly get it right!