Lecture 5: Linear systems and Intersecting Hyperplanes

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Lecture 4: Linear systems and intersecting (Hyper)planes

1. Orthogonality in $\mathbb{R}^n$

2. Geometric meaning of a linear equation

3. Intersecting planes in $\mathbb{R}^3$

Orthogonality in $\mathbb{R}^n$

Recall For a pair of (non-zero) vectors u and v in $\mathbb{R}^2$ or $\mathbb{R}^3$, the directed line segments that they represent are orthogonal (perpendicular) to each other ($u \perp v$) if and only if their scalar product $u \cdot v = 0$.

Definition For vectors $u = [a_1 \ a_2 \dots \ a_n]$ and $v = [b_1 \ b_2 \ \dots b_n]$ in $\mathbb{R}^n$, their scalar product (or dot product) is the number defined by

$$u \cdot v = a_1 b_1 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

Example

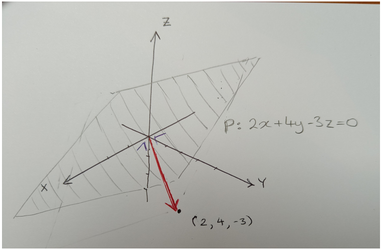
$[1 \ 3 \ -2] \perp [0 \ 4 \ 6]$ in $\mathbb{R}^3$, since $1(0) + (3)(4) + (-2)(6) = 0$.
$[5 \ 1 \ -2 \ 2]$ and $[3 \ -1 \ 9 \ 2]$ are orthogonal in $\mathbb{R}^4$.

Although the geometry is harder to visualize, the scalar product gives us an algebraic concept of orthogonality in $\mathbb{R}^n$.

Geometric meaning of a linear equation

Question What does the subset P of $\mathbb{R}^3$ consisting of all points whose coordinates satisfy $2x + 4y - 3z = 0$ look like?

$$
\begin{aligned}
& 2x + 4y - 3z = 0 \\
\iff \quad & [2\ 4\ \ -3] \cdot [x\ y\ z] = 0 \\
\iff \quad & [x\ y\ z] \perp [2\ 4\ \ -3]
\end{aligned}
$$



So P consists of all points whose coordinates represent vectors orthogonal to $[2\ 4\ \ -3]$. This is a plane, it looks like a copy of $\mathbb{R}^2$ inside $\mathbb{R}^3$, including the origin, and oriented so that it is perpendicular to its normal vector $[2\ 4\ \ -3]$.

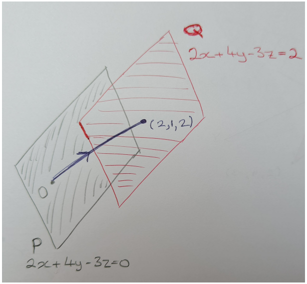
The points in $\mathbb{R}^n$ whose coordinates satisfy $a_1x_1 + a_2x_2 + \dots a_nx_n = 0$ are the vectors orthogonal to $n = [a_1, a_2, \dots, a_n]$. They form a hyperplane in $\mathbb{R}^n$, also called the orthogonal complement of n and denoted $n^\perp$. This "looks like" a copy of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$, passing through the origin.

What if the right hand side is not zero?

What about the set Q of points in $\mathbb{R}^3$ whose coordinates satisfy $2x + 4y - 3x = 2$? What does it look like?

We can write an example of such a point, [1 0 0] is one. Or [2 1 2].

$$
\begin{aligned}
[x\ y\ z] \in Q \quad &\Longleftrightarrow \quad [2\ 4\ {-3}] \cdot [x\ y\ z] = 2 = [2\ 4\ {-3}] \cdot [2\ 1\ 2] \\
&\Longleftrightarrow \quad [2\ 4\ {-3}] \cdot ([x\ y\ z] - [2\ 1\ 2]) = 0 \\
&\Longleftrightarrow \quad [2\ 4\ {-3}] \perp ([x\ y\ z] - [2\ 1\ 2]) = 0 \\
&\Longleftrightarrow \quad Q = [2\ 1\ 2] + P,
\end{aligned}
$$



Where P is the plane with equation $2x + 4y - 3x = 0$.

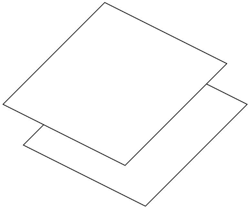
Conclusion: Q is obtained from P by adding [2 1 2] (or any vector that belongs to Q) to every vector in P. This means Q is the plane parallel to P that passes through the point $(2, 1, 2)$.

Intersecting planes in $\mathbb{R}^3$

Think about a system of linear equations in the variables x, y, z.
The set of points that satisfy any one equation is a plane in $\mathbb{R}^3$.
Solutions of the system are points that belong to all the planes. A pair of planes in $\mathbb{R}^3$ can intersect in three ways.

1. No intersection,

$$\begin{array}{rcrcrcl} 2x & + & y & - & 3z & = & 1 \\ 2x & + & y & - & 3z & = & 2 \end{array}$$

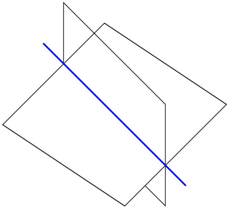


2. The two planes are identical,

$$\begin{array}{rcrcrcl} 2x & + & y & - & 3z & = & 1 \\ 4x & + & 2y & - & 6z & = & 2 \end{array}$$

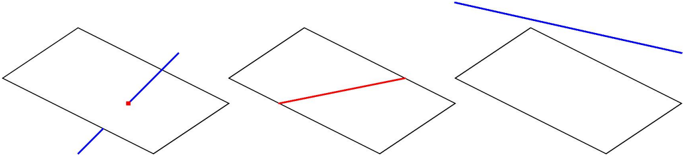
3. The two planes intersect in a line, (the "expected" situation)

$$\begin{array}{rcrcrcl} x & + & y & - & 3z & = & 1 \\ x & - & y & + & 2z & = & 2 \end{array}$$



Intersection of a plane and a line in $\mathbb{R}^3$

Given a line and a plane in $\mathbb{R}^3$, we “expect” that they intersect in one point. But the plane can contain the line, or they can have no point in common.

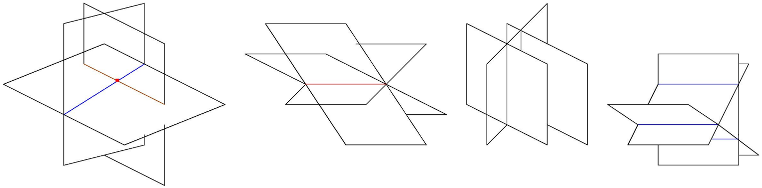


If n is a normal vector to the plane and v is a vector along the line, then

- $n \cdot v \neq 0$ in the first case.
- $n \cdot v = 0$ in the second and third cases.

Intersection of three planes in $\mathbb{R}^3$

Given three planes in $\mathbb{R}^3$, the "expected" situation is that the first two intersect in a line and this line interesects the third in a single point. So there is one point belonging to all three.
But the three planes could all intersect in a line. Or they could have empty intersection, if two of them are parallel, or if each pair intersects in a line but these three lines are parallel.



The system consisting of the equations of the three planes has

- A unique solution in the first case.
- One free variable in the second case, and infinitely many solutions.
- No solution in the third or fourth case.

A system of equations in n variables

Each equation represents a hyperplane in $\mathbb{R}^n$. The "expected" situation is

- that the system consisting of the first two equations is consistent with $n-2$ free variables - but it could be inconsistent, or it could be that the first two equations represent the same hyperplane
- that the addition of the next equation (at each step) reduces the number of free variables by one, until we have a unique solution after n equations. But at any stage, the addition of the next equation could introduce an inconsistency, or it could be that every simultaneous solution so far is also a solution of the next equation.
- If there are more eqautions than variables, we expect that there is no solution, but there might be.

Exercise Write down examples of systems (with three or four variables) where each of these possibilities occurs.

Another look at the example from Lecture 3

(a) Find the general solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \end{array}$$

Solution: $(x_1, x_2, x_3, x_4) = (-1, 1, 2, 0) + t(-1, 2, -4, 1)$, a line in $\mathbb{R}^4$

(c) Find the unique solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 9 \end{array}$$

Solution: $(x_1, x_2, x_3, x_4) = (0, -1, 6, -1)$.
Adding the fourth equation specifies one point from the line of simultaneous solutions to the first three.