Lecture 2: How to present the solution to a linear system

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1. An Example

2. Important terminology
   - Reduced row echelon form (RREF)
   - Leading variables and free variables

3. Writing the solution to a system with free variables

Example: Solve the following linear system

$$\boxed{\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array}}$$

$$\begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \xrightarrow[R3 \to R3 - 2R1]{R2 \to R2 - 3R1} \begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 0 & -10 & -20 & 40 & -10 \\ 0 & -9 & -18 & 36 & -9 \end{bmatrix}$$

$$\xrightarrow{R2 \times (-\frac{1}{10})} \begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & -9 & -18 & 36 & -9 \end{bmatrix} \xrightarrow{R3 \to R3 + 9R2} \underbrace{\begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{REF}}$$

$$\xrightarrow{R1 \to R1 - 3R2} \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF ✓ (Reduced row echelon form)

How to read the solution from the RREF

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & RHS \end{array} \end{array}$$

$$\begin{array}{lcccccccccc} \text{Equation 1:} & \mathbf{x_1} & & & - & x_3 & + & 3x_4 & = & 2 \\ \text{Equation 2:} & & \mathbf{x_2} & + & & 2x_3 & - & 4x_4 & = & 1 \\ \text{Equation 3:} & (& & & & & & 0 & = & 0) \end{array}$$

- Equation 1 says $x_1 = 2 + x_3 - 3x_4$
- Equation 2 says $x_2 = 1 - 2x_3 + 4x_4$
- Equation 3 has no content (an unexpected feature of this example).

A solution of the system must only satisfy $x_1 = 2 + x_3 - 3x_4$ and $x_2 = 1 - 2x_3 + 4x_4$, with no restriction on the values of x_3 and x_4.

We write s and t for the values of x_3 and x_4 in a solution of the system.

$$\boxed{\text{General Solution } (x_1, x_2, x_3, x_4) = (2 + s - 3t, 1 - 2s + 4t, s, t), \ s, t \in \mathbb{R}}$$

$(x_1, x_2, x_3, x_4) = (2 + s - 3t, 1 - 2s + 4t, s, t),\ s, t \in \mathbb{R}$

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & RHS \end{array} \end{array}$$

$$\begin{array}{lrcrcrcrcr} \text{Equation 1:} & \mathbf{x_1} & & & - & x_3 & + & 2x_4 & = & 2 \\ \text{Equation 2:} & & & \mathbf{x_2} & + & 2x_3 & - & 4x_4 & = & 1 \\ \text{Equation 3:} & (& & & & & & 0 & = & 0) \end{array}$$

The general solution is a concise description of all of the infinitely many solutions of the simplified system, and the original one. Particular solutions arise by specifying s and t in the general solution. For example

- $s = 0, t = 0$ $(x_1, x_2, x_3, x_4) = (2 + 0 - 0, 1 - 0 + 0, 0, 0) = (2, 1, 0, 0)$
- $s = 1, t = 0$ $(x_1, x_2, x_3, x_4) = (2 + 1 - 0, 1 - 2 + 0, 1, 0) = (3, -1, 1, 0)$
- $s = 5, t = 10$ $(x_1, x_2, x_3, x_4) = (2 + 5 - 30, 1 - 10 + 40, 5, 10) = (-23, 31, 5, 10)$

Exercise Check that these (and more examples that you write down) are solutions of the original system:

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array}$$

Some vocabulary and definitions

$$\begin{bmatrix} 1 & 3 & 5 & -9 & 5 \\ 3 & -1 & -5 & 13 & 5 \\ 2 & -3 & -8 & 18 & 1 \end{bmatrix} \xrightarrow[Gauss-Jordan \;\; elimination]{Elementary \;\; Row \;\; Operations} \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The second matrix is in reduced row echelon form (RREF). This means

1. Every row that is not all zeros has 1 as its first non-zero entry.
2. Every column that contains a leading 1 has 0 in all other positions.
3. The leading 1s go left to right as we move down through the rows.
4. Any rows that are all zeros are at the bottom.

Remark A matrix is in row echelon form has the above properties *except* that leading 1's can have non-zero entries above them in their columns.

Example This matrix is in REF, not RREF.
Every RREF is a REF.
Not every REF is a RREF.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 4 & -2 \\ 0 & 1 & 5 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 20 \end{bmatrix}$$

Leading variables and free variables

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array} \longrightarrow \begin{array}{c} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & RHS \end{array} \end{array}$$

Leading 1s in the RREF occur in the columns of the variables x_1 and x_2. The columns of x_3 and x_4 do not contain leading 1s.

Definition Any variables whose columns in the RREF do not contain leading 1s are called free variables. Variables whose columns do contain leading 1s are called leading variables.

In this example, x_1 and x_2 are leading variables, x_3 and x_4 are free.
In a solution, a free variable can take on any value.
Each non-zero row of the RREF describes how the value of one leading variable depends on those of the free variables.[1]

[1] See notes on inconsistent systems later

How to write the solution

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & - & 9x_4 & = & 5 \\ 3x_1 & - & x_2 & - & 5x_3 & + & 13x_4 & = & 5 \\ 2x_1 & - & 3x_2 & - & 8x_3 & + & 18x_4 & = & 1 \end{array} \longrightarrow \begin{array}{c} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & RHS \end{array} \end{array}$$

1. Give independent parameter names to the values of the free variables in a solution. Write (something like) *The values of the free variables x_3 and x_4 in a solution of the system are denoted s and t.*
2. Read, from the RREF, how the corresponding values of the leading variables depend on s and t. State the general solution

   $$(x_1, x_2, x_3, x_4) = (2 + s - 3t, 1 - 2s + 4t, s, t), \ \ s, t \in \mathbb{R}$$

   Note: the last part ($s, t \in \mathbb{R}$) is an essential (not optional) part of the statement. It tells the reader what values s and t may have.
3. Alternative version of the statement (either form is fine):

   $$(x_1, x_2, x_3, x_4) = (2, 1, 0, 0) + s(1, -2, 1, 0) + t(-3, 4, 0, 1), \ \ s, t \in \mathbb{R}.$$

   This version has a geometric meaning - more on that in Lecture 4.