Definition Let S be a subset of a vector space V, having at least 2 elements. Then S is *linearly independent* if no element of S is a linear combination of the other elements of S (equivalently, if no element of S belongs to the span of the other elements of S).

A subset consisting of a single element is linear independent, provided that its unique element is not the zero vector.

To decide if a given set is linearly independent, the above definition is not always the most useful formulation. The following altenative version is often more useful in practice.

Definition (Equivalent version) Let S be a non-empty subset of a vector space V. Then S is *linearly independent* if the only way to write the zero vector in V as a linear combination of elements of S is to take all the coefficients to be 0.

Equivalence of the two definitions

Let $S = \{v_1, \dots, v_k\}$ and suppose that $v_1 \in \langle v_2, \dots, v_k \rangle$. Then

$$v_1 = a_2 v_2 + \cdots + a_k v_k,$$

and

$$0_V = -v_1 + a_2 v_2 + \cdots + a_k v_k$$

is an expression for the zero vector as a linear combination of elements of S, whose coefficients are not all zero.

On the other hand, suppose that

$$0 = c_1 v_1 + \cdots + c_k v_k$$

where the scalars c_i are not all zero. If $c_1 \neq 0$ (for example), then the above equation can be rearranged to express v_1 as a linear combination of $v_2, \dots, v_k$:

$$v_1 = -\frac{c_2}{c_1} v_2 - \cdots - \frac{c_k}{c_1} v_k$$

An example in $\mathbb{R}^3$

In $\mathbb{R}^3$, let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} \right\}$.
To determine whether S is linearly independent, we must investigate whether the system of equations

$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solutions other than $(x, y, z) = (0, 0, 0)$. The augmented matrix of this system, and its RREF, are

$$\begin{bmatrix} 1 & -2 & -3 & 0 \\ 2 & 3 & 8 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An example in $\mathbb{R}^3$

Thus for any t, $(x, y, z) = (-t, -2t, t)$ is a solution, and for example by taking $t = 1$ we see that

$$-1\begin{bmatrix}1\\2\\-1\end{bmatrix} - 2\begin{bmatrix}-2\\3\\2\end{bmatrix} + 1\begin{bmatrix}-3\\8\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

and hence that each of the three elements of S is a linear combination of the other two. So S is not linearly independent (we say that S is *linearly dependent*).

Characterizations of linearly independent sets

Let S be a subset of a vector space V.

1. S is linearly independent if S is a *minimal* spanning set of its linear span - no proper subset of S spans the same subspace of V that S does.
2. S is linearly independent if every element of $\langle S \rangle$ has a *unique* expression as a linear combination of elements of S.
3. Another version of 2. above: S is linearly independent if every element of the span of S has unique coordinates in terms of the elements of S.

So a linearly independent set in a vector space V is a minimal or irredundant spanning set for its linear span. If its linear span happens to be all of V, it gets a special name.

> A *basis* of a vector space V is a spanning set of V that is linearly independent.

Lecture 12: Bases and dimension

Definition A *basis* of a vector space V is a spanning set of V that is linearly independent. [Plural: bases]

Lemma 10

If S is a finite spanning set of a vector space V, then S contains a basis of V.

Proof.

If S is not linearly independent, then some element v_1 of S is in the span of the other elements of S, and $S_1 := S \backslash \{v_1\}$ is again a spanning set of V. If S_1 is not linearly independent, then we can discard an element of S_1 that is in the linear span of the others, to form a smaller spanning set S_2. Since S is finite, this process cannot continue indefinitely, and it concludes with a linearly independent spanning set of V. □

The number of elements in a basis

We will show that if V has a finite basis, then *every* basis has the same number of elements. This number is then referred to as the *dimension* of V. The key to this is to show that the number of elements in *any* spanning set of V is an upper bound for the number of elements in *any* linearly independent subset of V.

Theorem

[Steinitz exchange lemma] Let V be a vector space, and suppose that $S = \{v_1, \dots, v_n\}$ is a spanning set of V. Then the number of elements in a linearly independent subset of V cannot exceed n.

Proof Outline Let $L = \{y_1, \dots, y_k\}$ be a linearly independent subset of V. We need to show $k \leq t$.

Proof outline for exchange lemma

Spanning set $S = \{v_1, \dots, v_n\}$. Linearly independent set $L = \{y_1, \dots, y_k\}$. Need to show $k \leq n$.

1. y_1 can be written as a linear combination of elements of S.
2. (After reordering) we can assume v_1 has a non-zero coefficient in such a combination.
3. Replace v_1 with y_1 to make $S_1 = \{y_1, v_2, \dots, v_n\}$. Argue that S_1 is still a spanning set.
4. y_2 is a combination of elements of S_1, not only involving y_1 but involving at least one of $v_1, \dots, v_n$ (say v_2). Replace v_2 with y_2 to get $S_2 = \{y_1, y_2, v_3, \dots, v_n\}$, another spanning set.
5. Keep going. If $k > n$, then after n steps we find that $\{y_1, \dots, y_n\}$ is a spanning set, hence y_{n+1} is a linear combination of these. Contradiction to the linear independence of L.

Consequences of the exchange lemma

Theorem

If V is a finite dimensional vector space over a field $\mathbb{F}$, then every basis of V has the same number of elements.

Proof.

Let B_1 and B_2 be bases of V. Then B_1 is linearly independent and B_2 is a spanning set of V, so $|B_1| \leq |B_2|$ by Theorem 61. Also, B_2 is linearly independent and B_1 is a spanning set of V, so $|B_2| \leq |B_1|$ by Theorem 61. Hence $|B_1| = |B_2|$. □

Definition The number of elements in any (hence every) basis of a finite dimensional vector space V is called the *dimension* of V, denoted $\dim V$.

An Example

Let V be the space of skew-symmetric matrices in $M_3(\mathbb{R})$ (a matrix A is *skew-symmetric* if $A^T = -A$). Then

$$V = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The typical element of V noted above can be written as

$$\begin{aligned} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} &= a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & a & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= a(E_{12} - E_{21}) + b(E_{13} - E_{31}) + c(E_{23} - E_{32}), \end{aligned}$$

where E_{ij} is the matrix with 1 in the (i, j)-position and zeros elsewhere. We see that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a spanning set of V. This set is also linearly independent. We conclude that $\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\}$ is a basis of V and that $\dim V = 3$.