We think of the real number line $\mathbb{R}$ as begin "1-dimensional", and of $\mathbb{R}^2$ as being "2-dimensional" and of $\mathbb{R}^3$ as being 3-dimensional. These terms are used not only in mathematics but in everyday language aswell. In linear algebra, they mean something quite precise.

To say that $\mathbb{R}$ is 1-dimensional means that we only need one real number to specify the position of a point in $\mathbb{R}$.

For a point in $\mathbb{R}^2$, we need to specify two real numbers, for example its x and y coordinates - but these are not the only options. We could specify its position relative to another pair of lines, instead of the two coordinate axes.

Another example of a vector space that is 2-dimensional is the space V consisting of all symmetric 2×2 matrices in $M_2(\mathbb{R})$ with trace zero. A symmetric matrix is one that is equal to its transpose. Trace zero means the sum of the entries on the main diagonal is zero.

Vector space - the "official" definition

Definition A vector space V over a field $\mathbb{F}$ is a non-empty set equipped with an addition operation $(+)$, and whose elements can be multiplied by scalars in $\mathbb{F}$, subject to the following axioms.

1. For all $u, v \in V$, $u + v = v + u$ (addition is commutative).
2. For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$ ($+$ is associative).
3. V has an element 0_V, with $0_V + v = v$ for all $v \in V$ (zero element).
4. For every $v \in V$, there exists an element $-v$ of V, with the property that $v + (-v) = 0_V$ (subtraction).
5. If $\alpha, \beta \in \mathbb{F}$ and $v \in V$, then $\alpha(\beta v) = \alpha\beta(v)$ (compatibility of scalar multiplication with multiplication in $\mathbb{F}$).
6. If $\alpha \in \mathbb{F}$ and $u, v \in V$, then $\alpha(u + v) = \alpha u + \alpha v$ (distributivity of scalar multiplication over addition in V).
7. If $\alpha, \beta \in \mathbb{F}$ and $v \in V$, then $(\alpha + \beta)v = \alpha v + \beta v$ (distributivity of scalar multiplication over addition in $\mathbb{F}$).
8. $1_{\mathbb{F}} v = v$ for all $v \in V$, where $1_{\mathbb{F}}$ is the multplicative identity element of $\mathbb{F}$.

Examples (and non-examples) of Vector Spaces

1. The set $\mathbb{Q}[x]$ of all polynomials with rational coefficients is a vector space over $\mathbb{Q}$.
2. The set $M_{2\times 3}(\mathbb{R})$ of all 2×3 matrices with real entries is a vector space over $\mathbb{R}$.
3. $\mathbb{C}$ is a vector space over $\mathbb{R}$, also over $\mathbb{Q}$.
4. The set of *all* matrices with real entries is not a vector space (since not all pairs of matrices can be added together).
5. The set of all 3×3 matrices with real entries and non-zero determinant is not a vector space.

Subspaces

Definition Let V be a vector space over $\mathbb{R}$. A subset U of V is a subspace (or vector subspace) of V if U is itself a vector space over $\mathbb{F}$, under the addition and scalar multiplication operations of V.

Two things need to be checked to confirm that a subset U of a vector space V is a *subspace*:

1. That U is *closed* under the addition in V: that $u_1 + u_2 \in U$ whenever $u_1 \in U$ and $u_2 \in U$;
2. That U is *closed* under scalar multiplication: that $\alpha u \in U$ whenever $u \in U$ and $\alpha \in \mathbb{F}$.

Examples of Subspaces

1. Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subset consisting of all polynomials of degree at most 2. This means that $P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}$. Then P_2 is a *(vector) subspace* of $\mathbb{Q}[x]$. If $f(x)$ and $g(x)$ are rational polynomials of degree at most 2, then so also is $f(x) + g(x)$. If $f(x)$ is a rational polynomial of degree at least 2, then so is $\alpha f(x)$ for any $\alpha \in \mathbb{Q}$.
2. The set of $\mathbb{C}$ complex numbers is a vector space over the set of real numbers. Within $\mathbb{C}$, the subset $\mathbb{R}$ is an example of a vector subspace over $\mathbb{R}$. An example of a subset of $\mathbb{C}$ that is *not* a real vector subset is the unit circle S in the complex plane - this is the set of complex numbers of modulus 1, it consists of all complex numbers of the form $a + bi$, where $a^2 + b^2 = 1$. This is closed neither under additon nor multiplication by real scalars.

Examples of Subspaces

3. The Cartesian plane $\mathbb{R}^2$ is a real vector space. Within $\mathbb{R}^2$, let $U = \{(a, b) : a \geq 0, b \geq 0\}$. Then U is closed under addition and under multiplication by positive scalars. It is not a vector subspace of $\mathbb{R}^2$, because it is not closed under multiplication by negative scalars.
4. Let v be a (fixed) non-zero vector in $\mathbb{R}^3$, and let

$$v^\perp = \{u \in \mathbb{R}^3 : u^T v = 0\}.$$

Then $v^\perp$ is not empty since $0 \in v^\perp$. Suppose that $u_1, u_2 \in v^\perp$. Then

$$(u_1 + u_2)^T v = (u_1^T + u_2^T)v = u_1^T v + u_2^T v = 0.$$

So $u_1 + u_2 \in v^\perp$ and $v^\perp$ is closed under addition.
If $u \in v^\perp$ and $\alpha \in \mathbb{R}$, then $(\alpha u)^T v = \alpha u^T v = \alpha 0 = 0$, and $\alpha u \in v^\perp$. Hence $v^\perp$ is closed under scalar multiplication in $\mathbb{R}^3$.
Conclusion: $v^\perp$ is a vector subspace of $\mathbb{R}^3$. Note that $v^\perp$ is not all of $\mathbb{R}^3$, since $v \notin v^\perp$.

The linear span of a set

Definition Let V be a vector space over a field $\mathbb{F}$, and let S be a non-empty subset of V. The $\mathbb{F}$-*linear span* (or just *span*) of S, denoted $\langle span \rangle$ is the set of all $\mathbb{F}$-linear combinations of elements of S in V. If $S = V$, then S is called a *spanning set* of V. This means that every element of V is a linear combination of elements of S.

Lemma If S is a subset of a vector space V, then $\langle S \rangle$ is a subspace of V, and it is the smallest subspace of V that contains the set S.

Example

Let $\mathbb{Q}[x]$ be the set of all polynomials with rational coefficients. Within $\mathbb{Q}[x]$, let P_2 be the subspace consisting of all polynomials of degree at most 2,

$$P_2 = \{a_2x^2 + a_1x + a_0 : a_0, a_1, a_2 \in \mathbb{Q}\}.$$

If $S = \{x^2 + 1, x + 1\}$, then

$$\langle S \rangle = \{a(x^2 + 1) + b(x + 1) : a, b \in \mathbb{Q}\} = \{ax^2 + bx + a + b : a, b \in \mathbb{Q}\}.$$

So $\langle S \rangle$ consists of all rational polynomials of degree at most 2, in which the constant coefficient is the sum of the coefficients of x and x^2. For example, $x^2 + 2x + 3 \in \langle S \rangle$ but $x^2 + 2x + 4 \notin \langle S \rangle$.

An example in $\mathbb{R}^2$

The set $S = \left\{ \binom{3}{1}, \ \binom{2}{1}, \ \binom{1}{-1} \right\}$ is a spanning set of the vector space $\mathbb{R}^2$ of all real column vectors with two entries. If $v = \binom{a}{b} \in \mathbb{R}^2$, we can write v as a linear combination of the elements of S, for example by writing

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a+b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-a-b)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - b\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This is not the only way to do it. We could also write

$$\begin{pmatrix} a \\ b \end{pmatrix} = (4a+b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-5a-b)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We could forget about the third element of S and just write

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a-2b)\begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-a+3b)\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

So all three elements of S are not needed to span $\mathbb{R}^2$. We could do it just with the subset $\{\binom{2}{1}, \binom{3}{1}\}$. Note that $\binom{1}{-1}$ is a $\mathbb{R}$-linear combination of the other two elements of S. If we drop this element from S, we can still recover it in the span of the remaining elements.

Finite dimensional and infinite dimensional spaces

Lemma 8

Suppose that $S_1 \subset S$*, where* S *is a subset of a vector space* V*. Then* $\langle S_1 \rangle \subseteq \langle S \rangle$*, and* $\langle S_1 \rangle = \langle S \rangle$ *if and only if every element of* $S \backslash S_1$ *is a linear combination of elements of* S_1*.*

We finish this section by noting the distinction between a *finite dimensional* and *infinite dimensional* vector space.

Definition 9

A vector space is said to be *finite dimensional* if it has a finite spanning set. A vector space that does not have a finite spanning set is *infinite dimensional*.

Two examples of infinite dimensional vector spaces

1. The vector space $\mathbb{R}[x]$ of all polynomials with real coefficients is infinite dimensional. To see this, let S be any finite subset of $\mathbb{R}[x]$ (i.e. a finite set of polynomials). Let x^k be the highest power of x to appear in any element of S. Then no linear combination of elements of S has degree exceeding k, so the linear span of S cannot be all of $\mathbb{R}[x]$.
2. The set $\mathbb{R}$ of real numbers is infinite dimensional as a vector space over the field $\mathbb{Q}$ of rational numbers.